(1) (40 pts) For this problem, assume $A$ is a $4 \times 4$ matrix with:

$$\text{range}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

(a) (10 pts) Find an orthonormal basis for $\text{range}(A)$.

(b) (10 pts) Find $\text{rank}(A^T)$, $\text{dim}(\ker(A^T))$, $\text{dim}(\text{coker}(A^T))$, $\text{dim}(\text{range}(A^T))$, and $\text{dim}(\text{corange}(A^T))$.

(c) (10 pts) Find an orthogonal basis for $\text{coker}(A)$. Show all work.

(d) (10 pts) Would $Ax = b$ have a solution if $b = (-1, 2, -1, -1)^T$? What if $b = (-2, 3, 2, -3)^T$? Explain.

**Solution:**

(a) Using the Gram-Schmidt process, let $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ and let

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

and now normalizing yields the orthonormal basis $\{u_1, u_2\} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\}$.

(b) From the Fundamental Theorem of Linear Algebra we know $\text{rank}(A^T) = \text{rank}(A)$ and $\text{rank}(A) = \text{dim}(\text{range}(A))$, and from part (a) $\text{dim}(\text{range}(A)) = 2$, so, by the Fundamental Theorem of Linear Algebra for a $4 \times 4$ matrix, we have,

$$\text{dim}(\text{range}(A^T)) = \text{dim}(\text{corange}(A^T)) = \text{rank}(A^T) = 2$$
and, $\text{dim}(\ker(A^T)) = 4 - 2 = 2$, and, $\text{dim}(\text{coker}(A^T)) = 4 - 2 = 2$.

(c) Note that $\text{coker}(A) = \{y | A^T y = 0\}$, so we need all vectors $y$ which are orthogonal to the columns of $A$ with respect to the dot product, and the range of $A$ is a basis for the columns of $A$, so we need

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \cdot y = 0 \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \cdot y = 0$$

that is, we need to solve,

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 1 & -1 \end{pmatrix} y = 0 \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} y = 0 \rightarrow \begin{pmatrix} y_1 - y_3 = 0 \\ y_2 - y_4 = 0 \\ y_3 = y_3 \\ y_4 = y_4 \end{pmatrix} \rightarrow \begin{pmatrix} y_1 = y_3 \\ y_2 = y_4 \end{pmatrix}.$$

so, $y = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} y_3 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} y_4$, and so, $\text{coker}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$. Note that this basis is already orthogonal.
(d) The system is consistent if \( b \in \text{range}(A) \), so the system is consistent if

\[
b = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}
\]

has a solution. So if \( b = (-1, 2, -1, -1)^T \) we have

\[
\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \\ 0 & -1 & -1 \end{pmatrix} \Rightarrow \text{system is inconsistent} \Rightarrow Ax = b \text{ does not have a solution.}
\]

If \( b = (-2, 3, 2, -3)^T \) we have

\[
\begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ -1 & 1 & 2 \\ 0 & -1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{system is consistent} \Rightarrow Ax = b \text{ has a solution.}
\]

(2) (40 pts) Show all work and justify your answers:

(a) (10 pts) Find the Gram matrix of \( F = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \). Is it positive definite? Justify your answer.

(b) (10 pts) Is the matrix \( B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \) positive definite? Justify your answer.

(c) (10 pts) Let \( v_1, v_2, v_3, v_4 \) be vectors in \( \mathbb{R}^3 \), can their Gram matrix be positive definite? Why or why not?

(d) (10 pts) Is the expression \( \langle u, v \rangle = u_1v_1 + u_1v_2 + u_2v_2 \) an inner product for vectors in \( \mathbb{R}^2 \)? Why or why not?

**Solution: (a)** The Gram matrix of \( F \) is \( K = F^T F = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \) and \( K \) is positive definite since the columns of \( F \) are linearly independent.

(b) Note that

\[
B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}
\]

so \( M \) is symmetric and regular with positive pivots and is therefore positive definite.

(c) No, since 4 vectors in \( \mathbb{R}^3 \) are linearly dependent (by a dimension argument) and so the corresponding Gram matrix cannot be positive definite, it is only positive semidefinite.

(d) Not an inner product since symmetry fails, for example, \( \langle (1, 0)^T, (1, 1)^T \rangle = 2 \neq 1 = \langle (1, 1)^T, (1, 0)^T \rangle \). It is positive and bilinear however.

(3) (20 pts) Find the least squares solution to the linear system:

\[
\begin{pmatrix} 3 & -1 \\ 0 & 2 \\ -2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}
\]

**Solution:** Let \( K = A^T A = \begin{pmatrix} 14 & 0 \\ 0 & 31 \end{pmatrix} \) and let \( f = A^T b = \begin{pmatrix} 9 \\ 4 \end{pmatrix} \) and solving \( Kx = f \) yields the least squares solution \( x^* = \begin{pmatrix} 9/14 \\ 4/31 \end{pmatrix} \).