1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $Poisson(\lambda)$ distribution. Verify that $S = \sum_{i=1}^n X_i$ is a sufficient statistic from the definition of sufficiency. (ie: do not use the Factorization Criterion)

2. Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$.
   (a) If $\mu$ is unknown and $\sigma^2$ is known, show that $\sum_{i=1}^n X_i$ is sufficient for $\mu$. Argue then that $\bar{X}$ is also sufficient for $\mu$.
   (b) If $\mu$ is known and $\sigma^2$ is unknown, show that $\sum_{i=1}^n (X_i - \mu)^2$ is sufficient for $\sigma^2$.
   (c) If $\mu$ and $\sigma^2$ are both unknown, show that $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n X_i^2$ are jointly sufficient for $\mu$ and $\sigma^2$. Argue then that $\bar{X}$ and $S^2$ are also jointly sufficient for $\mu$ and $\sigma^2$.

3. Let’s see the Rao-Blackwell Theorem in action!
   Let $X_1, X_2, \ldots, X_n$ be a random sample from the Bernoulli distribution with parameter $p$.
   (a) Find a sufficient statistic for $p$.
   (b) Show that $I\{X_1=1\}$ is an unbiased estimator of $p$.
   (c) Compute $E[I\{X_1=1\}|S]$ where $S$ is your sufficient statistic from part (a).
   (d) Show that your answer to part (c) is still unbiased for $p$.
   (e) [required for 5520 only] Compare the variances of $\hat{p}_1 = I\{X_1=1\}$ and $\hat{p}_2 = E[I\{X_1=1\}|S]$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample form the exponential distribution with rate $\lambda$. Verify, from the definition, that $S = \sum X_i$ is a complete statistic.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample Beta distribution with parameters $a = \theta$ and $b = 1$.
   (a) Find the UMVUE for $1/\theta$.
   (b) Find the UMVUE for $(\theta/(\theta + 1))^n$.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Pareto($\gamma$) distribution.
   (a) Find the UMVUE for $1/\gamma$.
   (b) Find the UMVUE for $\gamma$.

7. [Required for 5520 Students Only] A statistic $S$ is said to be minimal sufficient for a given distribution if it is sufficient and is a function of every other set of sufficient statistics. In this problem we are going to prove the following Theorem under a certain assumption for simplicity.

   If $S$ is complete and sufficient, then $S$ is minimal sufficient.

To prove this, let $S$ be a complete and sufficient statistic and let $M$ be minimal sufficient. (You are probably wondering whether or not we can assume the existence of a minimal sufficient statistic. This is the simplifying assumption. For a more general proof of this Theorem that does not start with this assumption, look up the proof someday when you are bored. This Theorem is known as “Bahadur’s Theorem”.)
(a) Argue that the quantity $S - \mathbb{E}[S|M]$ can be thought of as a function of $S$ “only”, as opposed to a function of $S$ and $M$.

(b) Find $\mathbb{E}[S - \mathbb{E}[S|M]]$.

(c) Use (a) and (b) to show that $S = \mathbb{E}[S|M]$.

(d) Conclude that $S$ is a function of $M$.

(e) Explain why this shows that $S$ is minimal sufficient.