Ever Expanding Solutions to Old Analysis Prelim Questions 1

1. [Aug 15 Problem 1] Let \( \{x_\alpha | \alpha \in I\} \) be an indexed set in a Banach space \( \mathbb{X} \), where the index set \( I \) may be countable or uncountable. For each finite subset \( J \) of \( I \), we define the partial sum \( S_J \) by

\[
S_J = \sum_{\alpha \in J} x_\alpha.
\]

The unordered sum of the indexed set \( \{x_\alpha | \alpha \in I\} \) is said to converge unconditionally to \( x \) if for every \( \epsilon > 0 \), there is a finite subset \( J^\epsilon \subset I \) such that \( \|S_J - x\| < \epsilon \) for all finite subsets \( J \) that contain \( J^\epsilon \).

Consider the following types of convergence of a series (with a countable index) in the Banach space \( \mathbb{X} \).

(a) convergence
(b) unconditional convergence
(c) absolute convergence
(d) Cauchy convergence (in the usual sense; you do not need to consider the sense of Cauchy convergence for an unordered sum)

What are the causal relationships among these four types of convergence? Your answer should cover all possible pairwise relationships among the different types of convergence, and each relationship should be of the form \((f) \Rightarrow (g)\), \((g) \Rightarrow (f)\), \((f) \Leftrightarrow (g)\), or “no implication”. Briefly justify each, e.g., if your answer is \((f) \Rightarrow (g)\), in addition to proving the implication, you should also briefly justify why \((g) \Leftrightarrow (f)\) [using, for example, a counter-example].

Solution:

\[
\begin{align*}
\text{Note that we are supposed to consider Cauchy convergence in the usual sense but, just for the record, Cauchy convergence in the unordered sum sense means that for every } \epsilon > 0, \text{ there is a finite subset } J^\epsilon \subset I \text{ such that, for all finite subsets } J \text{ that contain } J^\epsilon \text{ and all finite subsets } K \subset J, \\
\| \sum_{\alpha \in J} x_\alpha - \sum_{\alpha \in K} x_\alpha \| < \epsilon.
\end{align*}
\]

Without loss of generality, we will take our countable index set to be \( \mathbb{N} \). We will use \( \sum_{n \in \mathbb{N}} x_n \) to denote the unordered sum and not necessarily \( \sum_{n=1}^{\infty} x_n \). Note that the unordered sum \( \sum_{n \in \mathbb{N}} x_n \) converges if and only if the ordered sum \( \sum_{n=1}^{\infty} x_n \) converges unconditionally.
• [(a) $\not\Rightarrow$ (b)]: Consider the Banach space $(\mathbb{R}, \| \cdot \|)$, where $\| \cdot \|$ is the Euclidean norm, and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. If you reorder the series so that you do not have continual telescoping, it will not converge. Therefore it does not converge unconditionally.

• [(a) $\not\Rightarrow$ (c)]: Again consider $\mathbb{R}$ with the Euclidean norm and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

• [(a) $\Leftrightarrow$ (d)]: Convergence and Cauchy convergence are equivalent in a Banach space.

• [(b) $\Rightarrow$ (a)]: Clearly unconditional convergence implies convergence.

• [(b) $\not\Rightarrow$ (c)]: Consider $((\ell^2)(\mathbb{N}), \| \cdot \|_2)$, and, in particular, the sequences

\[
x_1 = (1, 0, 0, 0, \ldots)
\]
\[
x_2 = (0, 1/2, 0, 0, \ldots)
\]
\[
x_3 = (0, 0, 1/3, 0, \ldots)
\]

Then $\sum_{n \in \mathbb{N}} x_n$ converges unconditionally to $x = (1, 1/2, 1/3, \ldots) \in \ell^2(\mathbb{N})$.

However, for each $n \geq 1$, $\|x_n\| = 1/n$, so

\[
\sum_{n=1}^{\infty} \|x_n\| = \sum_{n=1}^{\infty} \frac{1}{n},
\]

which diverges. Thus the series does not converge absolutely.

(Note: One can show that unconditional convergence actually does imply absolute convergence for a finite dimensional Banach space!)

• [(b) $\Rightarrow$ (d)]: Unconditional convergence implies convergence which in turn implies Cauchy convergence.

• [(c) $\Rightarrow$ (a)]: Absolute convergence clearly implies convergence.

• [(c) $\Rightarrow$ (b)]: Suppose that the series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent. (i.e. $\sum_{n=1}^{\infty} \|x_n\| < \infty$) Then the series converges to some $x$. (i.e. $\sum_{n=1}^{\infty} x_n = x$)

Let $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that

\[
\left\| \sum_{n=1}^{N} x_n - x \right\| < \varepsilon/2
\]

Consider the rearranged series $\sum_{n=1}^{\infty} x_{\sigma(n)}$ where $\sigma : \mathbb{N} \to \mathbb{N}$ is some bijection.

Let $M \geq \max\{n : \sigma(n) \leq N\}$. Then the sum $\sum_{m=1}^{M} x_{\sigma(m)}$ will include the terms $x_1, x_2, \ldots, x_N$.

Thus, $\sum_{m=1}^{M} x_{\sigma(m)} - \sum_{n=1}^{N} x_n$ is a sum of a finite number of terms $x_n$ with all indices $\geq N + 1$.

So,

\[
\left\| \sum_{m=1}^{M} x_{\sigma(m)} - \sum_{n=1}^{N} x_n \right\| \leq \left\| \sum_{n=N+1}^{\infty} x_n \right\| \leq \sum_{n=N+1}^{\infty} \|x_n\|
\]
which can be made $< \varepsilon/2$ for large enough $N$ since the series is absolutely convergent.

Finally,

$$\left| \sum_{m=1}^{M} x_{\sigma(m)} - x \right| \leq \left| \sum_{m=1}^{M} x_{\sigma(m)} - \sum_{n=1}^{N} x_n \right| + \left| \sum_{n=1}^{N} x_n - x \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for large enough $M$. This implies that the sum converges unconditionally.

• $[(c) \Rightarrow (d)]$: Absolute convergence implies convergence which in turn implies Cauchy convergence.

• $[(d) \not\Rightarrow (b)]$ The alternating harmonic series in the first bullet is Cauchy but does not converge conditionally.

• $[(d) \not\Rightarrow (c)]$ The alternating harmonic series in the first bullet is Cauchy but does not converge absolutely.