1. Note that
\[ ||f_n|| = ||f_n - f + f|| \leq ||f_n - f|| + ||f||. \]
Also,
\[ ||f|| = ||f - f_n + f_n|| \leq ||f_n - f|| + ||f_n||. \]
So
\[ ||f|| - ||f_n - f|| \leq ||f_n|| \leq ||f_n - f|| + ||f||. \]
Letting \( n \to \infty \), \( ||f_n - f|| \to 0 \) implies that
\[ \lim_{n \to \infty} ||f_n|| = ||f||. \]

2. Ug. This problem was on the review for Exam I where it was also not true. I wonder how many more times I can cut and paste the same wrong problem?
This space is not bounded. Just because \( ||(x_n)||_2 < \infty \) for all sequences does not mean we can find a single \( M < \infty \) where we can say that \( ||(x_n)||_2 \leq M \) for all square-summable sequences. Also, it is not closed. To see this, consider the sequence of sequences \((1, 0, 0, \ldots), (2, 0, 0, \ldots), (3, 0, 0, \ldots), \ldots\). Each one is square-summable but the limit is not. Thus, the space of square-summable sequences does not contain all of its limit points and so it is not closed.
Finally, it is not compact. Consider the sequence of sequences \((1, 0, 0, \ldots), (0, 1, 0, \ldots), (0, 0, 1, 0, \ldots)\). This sequence of sequences has no convergent subsequence under the norm induced by the given metric.

3. Define
\[ Tu(x) = \cos x + \frac{1}{5} \sin (u^2(y)) \, dy. \]
Note that \( T \) maps \( C([0, 1]) \) functions to \( C([0, 1]) \) functions. Since \( C([0, 1]) \) is complete, the contraction mapping theorem will apply.
Let us show that \( T \) is a contraction.
\[ ||T u - T v||_\infty = \sup_{0 \leq x \leq 1} |Tu(x) - Tv(x)| \]
\[ = \sup_{0 \leq x \leq 1} \left| \frac{1}{5} \int_0^x [\sin u^2(y) - \sin v^2(y)] \, dy \right| \]
\[ \leq \frac{1}{5} \sup_{0 \leq x \leq 1} \int_0^x |\sin u^2(y) - \sin v^2(y)| \, dy \]
By the mean value theorem, we know that there is some \( s \in [0, 1] \) such that
\[ \frac{\sin u - \sin v}{u - v} \leq \cos s \leq 1 \]
\[ |\sin u(t) - \sin v(t)| \leq |u(t) - v(t)|. \]

So, we have that
\[
||Tu - Tv||_\infty \leq \frac{1}{5} \sup_{0 \leq x \leq 1} \int_0^x |u^2(y) - v^2(y)| \, dy
\]
\[
= \frac{1}{5} \sup_{0 \leq x \leq 1} \int_0^x |u(y) + v(y)| \cdot |u(y) - v(y)| \, dy
\]
\[
\leq \frac{1}{5} \sup_{0 \leq x \leq 1} \int_0^x (|u(y)| + |v(y)|) \cdot |u(y) - v(y)| \, dy
\]

Since \( u \) and \( v \) are assumed to be continuous functions on the closed bounded interval \([0, 1]\), they are bounded on \([0, 1]\), so, for some \( M > 0 \),
\[
||Tu - Tv||_\infty \leq \frac{M}{5} \sup_{0 \leq x \leq 1} \int_0^x |u(y) - v(y)| \, dy
\]
\[
\leq \frac{M}{5} \int_0^1 |u(y) - v(y)| \, dy \leq \frac{M}{5} ||u - v||_\infty \int_0^1 \, dy
\]
\[
= \frac{M}{5} ||u - v||_\infty
\]

This may or may not be a contraction, depending on the value of \( M \), but, we are trying to show existence of a solution in \( C([0, 1]) \). So, let’s limit our search to the set of continuous functions on \([0, 1]\) that are bounded, in the uniform norm, by some constant \( M \) and then choose this \( M \) appropriately after the fact. In that bounding above when we pulled out bounds on \(|u(y)|\) and \(|v(y)|\), I will more carefully pull out an \( M \) for each of them to get
\[
||Tu - Tv||_\infty \leq \frac{2M}{5} ||u - v||_\infty
\]

If we take \( M < 5/2 \), we have a contraction mapping. However, for the contraction mapping theorem to apply, it must be a contraction on the entire space being considered. If we define \( C := \{ u \in C([0, 1]) : ||u||_\infty \leq M \} \), we then have a complete subspace (a closed subset of a complete space is complete). You should convince yourself over the winter break that \( M \) can be chosen so that \( T : C \rightarrow C \)!

Since a contraction mapping on a complete space has a unique fixed point \( u \) such that \( Tu = u \), we have shown the existence of a solution!

4. Here \( \dot{u}(t) = f(t, u) \) where \( f(t, u) = \sqrt{a^2(t) + u^2} \). Note that this function is globally Lipschitz
\[|f(t, u) - f(t, v)| = \sqrt{a^2 + u^2} - \sqrt{a^2 + v^2}\]
\[= \frac{|(a^2+u^2)-(a^2+v^2)|}{\sqrt{a^2+u^2}+\sqrt{a^2+v^2}}\]
\[= \frac{|u^2-v^2|}{\sqrt{a^2+u^2}+\sqrt{a^2+v^2}}\]
\[= \frac{|u+v|}{\sqrt{a^2+u^2}+\sqrt{a^2+v^2}} \cdot |u - v|\]
\[\leq \frac{|u|+|v|}{\sqrt{a^2+u^2}+\sqrt{a^2+v^2}} \cdot |u - v|\]
\[\leq |u - v|.
\]

Thus, the ODE has a unique global solution.

5. I am going to change the \(y\)'s to \(u\)'s to make everything a little cleaner and more “familiar”...

\[u''(x) = 1 - 2xu(x)\]
\[\downarrow\]
\[u'(y) = \int_1^y [1 - 2su(s)] \, ds + C_1\]
\[\downarrow\]
\[u(x) = \int_0^x \int_1^y [1 - 2su(s)] \, ds \, dy + C_1 x + C_2\]
\[u(0) = 0 \Rightarrow C_2 = 0, \text{ so}\]
\[u(x) = \int_0^x \int_1^y [1 - 2su(s)] \, ds \, dy + C_1 x\]

Integrating by parts with \(u = \int_1^y [1 - 2su(s)] \, ds\) and \(dv = dy\) gives

\[u(x) = \left[ y \int_1^y (1 - 2su(s)) \, ds \right]_{y=0}^{y=x} - \int_0^x y \left[ 1 - 2yu(y) \right] \, dy + C_1 x\]
\[= x \int_1^x [1 - 2su(s)] \, ds - \int_0^x y \, dy + 2 \int_0^x yu(y) \, dy + C_1 x\]
\[= x \int_x^1 [2su(s) - 1] \, ds - \int_0^x y \, dy + 2 \int_0^x y^2 u(y) \, dy + C_1 x\]

Note that \(u(1) = 0 \Rightarrow\)
\[C_1 = \int_0^1 y \, dy - 2 \int_0^1 y^2 u(y) \, dy.\]

After some manipulation, we can then write

\[u(x) = \int_0^1 g(x, y) [2x u(x) - 1] \, dy\]
where \( g(x, y) \) is the Green’s function

\[
g(x, y) = \begin{cases} 
  x(1 - y), & 0 \leq x \leq y \leq 1 \\
  y(1 - x), & 0 \leq y \leq x \leq 1 
\end{cases}
\]

6. This is a Fredholm integral equation. Let \( Ku(x) = \cos x + \lambda \int_0^\pi xy u(y) \, dy \) and let \( u_0 = 0 \). Then,

\[
u_1(x) = Ku_0(x) = \cos x
\]

\[
u_2(x) = Ku_1(x) = \cos x + \lambda \int_0^\pi xy \cos y \, dy = \cos x - 2\lambda x
\]

\[
u_3(x) = Ku_2(x) = \cos x + \lambda x \int_0^\pi y(\cos y - 2\lambda y) \, dy
\]

\[
u_3(x) = \cos x - 2\left(\lambda + \lambda^2 \frac{\pi^3}{3}\right)x
\]

\[
u_4(x) = Ku_3(x) = \cos x + \lambda x \int_0^\pi y \left[\cos y - 2\left(\lambda + \lambda^2 \frac{\pi^3}{3}\right)y\right] \, dy
\]

\[
u_4(x) = \cos x - 2\left(\lambda + \lambda^2 \frac{\pi^3}{3} + \lambda^3 \frac{\pi^6}{6}\right)x
\]

\[
u_5(x) = Ku_4(x) = \cos x + \lambda x \int_0^\pi y \left[\cos y - 2\left(\lambda + \lambda^2 \frac{\pi^3}{3} + \lambda^3 \frac{\pi^6}{6}\right)y\right] \, dy
\]

\[
u_5(x) = \cos x - 2\left(\lambda + \lambda^2 \frac{\pi^3}{3} + \lambda^3 \frac{\pi^6}{6} + \lambda^4 \frac{\pi^9}{9}\right)x
\]

etc...

\[
u_n(x) = \cos x - 2\lambda x \sum_{i=0}^{n} \left(\frac{\lambda \pi^3}{3}\right)^i
\]

Provided \(|\lambda| < 3/\pi^3\), this converges to

\[
u(x) = \cos x - 2\lambda x \frac{1}{1 - \frac{\lambda \pi^3}{3}}.
\]

7. (a) In the usual topology finite sets are closed, but there are some closed subsets that are not finite. Thus the usual topology has more closed subsets than the cofinite topology and hence more open subsets. So, the usual topology is stronger.

(b) There are four topologies on \( X = \{a, b\} \). They are the discrete, trivial, \( \{\emptyset, X, \{a\}\} \) and \( \{\emptyset, X, \{b\}\} \) topologies. The last two are homeomorphic.

(c) \( \text{int}(\mathbb{Q}) = \emptyset, \text{cl}(\mathbb{Q}) = \mathbb{R} \)

\( \text{int}(\mathbb{Q}) = \mathbb{Q}, \text{cl}(\mathbb{Q}) = \mathbb{Q} \)

\( \text{int}(\mathbb{Q}) = \emptyset, \text{cl}(\mathbb{Q}) = \mathbb{R} \)

\( \text{int}(\mathbb{Q}) = \emptyset, \text{cl}(\mathbb{Q}) = \mathbb{R} \)
\[ \text{int}(\mathbb{Q}) = \emptyset, \text{cl}(\mathbb{Q}) = \mathbb{Q} \]
\[ \text{int}(\mathbb{Q}) = \emptyset, \text{cl}(\mathbb{Q}) = \mathbb{R} \]

8. Since \( \mathbb{X} \) is Hausdorff, for every \( y \in K \), there exists a neighborhoods \( U_y \) of \( y \) and \( V_y \) of \( x \) that are disjoint. (Yes, I meant for the neighborhood of \( x \) to be denoted with a \( y \) subscript. As you change \( y \) values, you may have to adjust the neighborhood around \( x \) so that the two neighborhoods are disjoint.)

By the definition of a neighborhood in a topological space, for each \( U_y \), there exists an open set \( G_y \) such that \( y \in G_y \subseteq U_y \) and for each \( V_y \), there exists an open set \( D_y \) such that \( x \in D_y \subseteq V_y \).

Now since \( K \) is compact and since \( \bigcup_{y \in K} G_y \) is an open cover of \( K \), there exists a finite subcover \( \{G_1, G_2, \ldots, G_n\} \) of \( K \) where each \( G_i \in \{G_y : y \in K\} \).

Note that
\[ \bigcup_{i=1}^{n} G_i \text{ is open and contains } K \]
\[ V := \bigcap_{i=1}^{n} D_i \text{ where } D_i \text{ is the open set containing } x \text{ that corresponds to } G_i \text{ is open and disjoint from } G. \]
\[ V \text{ contains } x \]

Done!

9. Before begining, let’s prove this claim:

A compact suset of a Hausdorff space is closed.

Proof: Let \( \mathbb{X} \) be a Hausdorff space and let \( K \subseteq \mathbb{X} \) be compact. We will show that \( \mathbb{X} \setminus K \) is open.

- Take any \( x \in \mathbb{X} \setminus K \). For any \( y \in K \), \( \mathbb{X} \) Hausdorff implies that there exist disjoint neighborhoods \( U_x \) of \( x \) and \( V_y \) which, by definition, contain open sets \( G_x \) and \( G_y \) with \( x \in G_x \) and \( y \in G_u \).
- For every \( y \in K \), find an open set \( G_y \) containing \( y \) and a disjoint set \( G_{xy} \) containing \( x \).
- Note that \( \bigcup_{y \in K} G_y \) is an open cover of \( K \). Since \( K \) is compact, it contains a finite subcover \( \bigcup_{i=1}^{n} G_{yi} \supseteq K \).
- Also note that \( G := \bigcap_{i=1}^{n} G_{xy} \) is an open set that does not intersect the finite subcover of \( K \) and therefore does not intersect \( K \).
- So, \( x \in G \subseteq \mathbb{X} \setminus K \) which implies that \( \mathbb{X} \setminus K \) is open and hence that \( K \) is closed. \( \square \)

Consider \( \mathbb{X} \setminus U \) which is closed in \( \mathbb{X} \). Since a closed subset of a compact set is compact, we know that \( \mathbb{X} \setminus U \) is also compact.

\( f \) continuous and \( \mathbb{X} \setminus U \) compact \( \Rightarrow f(\mathbb{X} \setminus U) \) is compact in \( \mathbb{Y} \).
By a problem on take home exam 3, we know that a compact subset of a Hausdorff space is necessarily closed. Hence \( f(X \setminus U) \) is closed in \( Y \).

Let \( V = Y \setminus f(X \setminus U) \). Note that \( V \) is open in \( Y \). Note also that it contains \( C \).

Since \( X \setminus U \subseteq f^{-1}(f(X \setminus U)) \), we have

\[
 f^{-1}(V) = X \setminus f^{-1}(f(X \setminus U)) \subseteq X \setminus (X \setminus U) = U
\]

as desired.

(It would help to draw some pictures!)

10. This problem is cancelled from the review because we did not get as far as I wanted to in class yet. For a proof, see Section 5.4 in H & N. Don’t worry about it for this exam though.