13.7 Surface Integration

Suppose we want to find the surface area of a surface $S$ described by the function $g(x, y, z) = c$ where $c$ is a constant. Or, similarly, suppose we know that the density of a thin metal shell described by surface $S$ and we want to integrate over $S$ to find the mass. Both of these things require us to be able to integrate over surfaces. Consider the following picture.

We do surface integration in a similar fashion to every other form of integration we’ve done so far. We come up with a formula for the surface area of the little surface area element $\Delta \sigma$, evaluate the function on the little surface area chunk, add up the contribution from all the little chunks, and then take the limit as the number of chunks goes to infinity. In the end we have something that looks like

$$\int \int_S f(x, y, z) \, d\sigma$$

If we only want to compute the area of the surface then we perform the integral with $f(x, y, z) = 1$, or

$$SA = \int \int_S \, d\sigma$$

We compute surface integrals similar to the way we compute line integrals. With line integrals we had to write the little arc length element $ds$ in terms of a 1D element $dt$, i.e.

$$ds = |v| \, dt$$

where $|v|$ was the Jacobian of the transformation from the curved space to a line. Then we computed the line integral according to
\[
\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \, |v| \, dt
\]

For surface integrals we will write the surface area element \(d\sigma\) in \(S\) in terms of the area element \(dA\) in \(R\) obtained by projecting \(d\sigma\) onto one of the coordinate planes. The transformation looks like

\[
d\sigma = \frac{\left| \nabla g \right|}{\left| \nabla g \cdot p \right|} \, dA
\]

where \(g\) is the function defining the surface and \(p\) is the vector in the direction in which we project \(S\) onto the chosen coordinate plane. For example, in the picture above, imagine that the plane of projection is the \(xy\)-plane, then we project \(S\) along the vector \(p = \mathbf{k}\).

In general, we compute the surface integral according to

\[
\int \int_S f(x, y, z) \, d\sigma = \int \int_R f(x, y, z) \frac{\left| \nabla g \right|}{\left| \nabla g \cdot p \right|} \, dA
\]

The steps needed to compute the surface integral are as follows

1. Choose coordinate plane to project surface onto and define unit vector \(p\)
2. Determine region of projection \(R\)
3. Compute \(\left| \nabla g \right|\) and \(\left| \nabla g \cdot p \right|\) and construct integrand (Jacobian + Function)
4. Eliminate extra variables using the surface equation \(g(x, y, z) = c\)
5. Evaluate regular area integral over \(R\)

Example 1: Compute the surface area of the hemisphere of radius 1 above the \(xy\)-plane.

The equation of a sphere of radius 1 is given by \(g(x, y, z) = x^2 + y^2 + z^2 = 1\). If we draw the hemisphere we have
We choose to project the surface into the $xy$-plane. We make this choice because projecting into either of the other coordinate planes would cause the surface to fold over itself. This is not, strictly speaking, impossible, but it would require computing two surface integrals. For example, if we projected the surface into the $yz$-plane we would need to do the part of the surface defined for $x > 0$ separately from the part of the surface defined for $x < 0$.

If we project onto the $xy$-plane then we have $\mathbf{p} = k$.

The projected region $R$ is the circle described by $x^2 + y^2 = 1$.

To find the integrand, we need to compute $|\nabla g|$ and $|\nabla g \cdot \mathbf{p}|$

$$\nabla g = \langle 2x, 2y, 2z \rangle \Rightarrow |\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} \text{ and } |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot k| = |2z|$$

Note that since the surface lies above the $xy$-plane we do not need the absolute values around the $2z$ term. Since we’re finding the surface area of the hemisphere, the only thing in the integrand is the Jacobian

$$\frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} = \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z} = \frac{\sqrt{x^2 + y^2 + z^2}}{z}$$

Since we’ve projected the surface into the $xy$-plane we can’t have $z$ in the integrand. To eliminate the $z$ variables we use the equation of the surface: $x^2 + y^2 + z^2 = 1$

$$\frac{\sqrt{x^2 + y^2 + z^2}}{z} = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

We then have

$$\int \int_S d\sigma = \int \int_R \frac{1}{\sqrt{1 - x^2 - y^2}} dA$$

Since the region $R$ is the unit circle in the $xy$-plane we convert to polar coordinates. The integral then becomes

$$\int \int_S d\sigma = \int \int_R \frac{1}{\sqrt{1 - x^2 - y^2}} dA = \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{1 - r^2}} r \, dr \, d\theta = 2\pi$$

This agrees with our geometric intuition because the surface area of a unit sphere is $4\pi$ and here we are just finding the area of the top half.
Deriving the Jacobian of the Transformation

So far we’ve stated the transformation between $d\sigma$ and $dA$ as fact. Let’s see where it comes from. First we make the assumption that we can approximate the surface area element $\Delta \sigma$ by it’s tangent plane. This is reasonable since in the end we’re going to take the limit as the size of $\Delta \sigma \to 0$. In this limit the surface area element appears very flat and the approximation is valid. Consider the following picture

We now try to find a relationship between the area elements $\Delta \sigma_k$ and $\Delta A_k$. Since $\Delta \sigma_k$ is a rectangle, we can find it’s area by taking the magnitude of the cross-product of vectors $\mathbf{u}_k$ and $\mathbf{v}_k$

$$\Delta \sigma_k = |\mathbf{u}_k \times \mathbf{v}_k|$$

Note that this area is not the same as $\Delta A_k$ unless $\Delta A_k$ and $\Delta \sigma_k$ are parallel. To get the area of the projection we use

$$\Delta A_k = |(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}|$$

This should be believable because if $(\mathbf{u}_k \times \mathbf{v}_k)$ (which is parallel to $\nabla g$) is parallel to $\mathbf{p}$ then the quantity $\Delta A_k$ as defined above will be large. If $(\mathbf{u}_k \times \mathbf{v}_k)$ is not parallel to $\mathbf{p}$ then $\Delta A_k$ gets smaller. Then, by the definition of the dot product

$$\Delta A_k = |(\mathbf{u}_k \times \mathbf{v}_k) \cdot \mathbf{p}|$$

$$= |(\mathbf{u}_k \times \mathbf{v}_k)||\mathbf{p}||\cos(\gamma_k)|$$

$$= \Delta \sigma_k |\cos(\gamma_k)|$$

Solving this expression for $\Delta \sigma_k$ we have
\[ \Delta \sigma_k = \frac{\Delta A_k}{|\cos (\gamma_k)|} \]

Taking the limit as the size of the surface area elements goes to 0 we have

\[ d\sigma = \frac{dA}{|\cos (\gamma)|} \]

Now we need to compute \(|\cos (\gamma)|\) in terms of quantities that we know. We have

\[ |\nabla g \cdot p| = |\nabla g| |p| |\cos (\gamma)| \]
\[ = |\nabla g| |\cos (\gamma)| \]

Rearranging this expression we have

\[ \frac{1}{|\cos (\gamma)|} = \frac{|\nabla g|}{|\nabla g \cdot p|} \]

and finally

\[ d\sigma = \frac{|\nabla g|}{|\nabla g \cdot p|} dA \]

**Example 2:** Find the area of the paraboloid \( z = x^2 + y^2 \) cut by the plane \( z = 4 \).

To determine the function \( g(x, y, z) = c \) that describes the surface, we need to put all the variables on one side of the equals sign and all the constants on the other. Moving the \( z \) to the other side of the equation of the paraboloid we have

\[ g(x, y, z) = x^2 + y^2 - z \]

To help choose the direction of projection it is usually a good idea to plot the surface.
We again want to project the surface into the \(xy\)-plane, so we choose \(p = k\).

Since the paraboloid is cut by the plane \(z = 4\), the projected region \(R\) is given by

\[
z = x^2 + y^2 \quad \Rightarrow \quad 4 = x^2 + y^2
\]

which is a circle of radius 2.

To find the integrand, we need to compute \(|\nabla g|\) and \(|\nabla g \cdot p|\)

\[
\nabla g = (2x, 2y, -1) \quad \Rightarrow \quad |\nabla g| = \sqrt{4x^2 + 4y^2 + 1} \quad \text{and} \quad |\nabla g \cdot p| = |\nabla g \cdot k| = |-1| = 1
\]

Then we have

\[
\frac{|\nabla g|}{|\nabla g \cdot p|} = \frac{\sqrt{4x^2 + 4y^2 + 1}}{1} = \sqrt{4x^2 + 4y^2 + 1}
\]

Note that this time \(z\) does not appear in the integrand, so we don’t need to eliminate any variables. We then have

\[
\iint_S d\sigma = \iint_R \sqrt{4x^2 + 4y^2 + 1} \, dA
\]

Since \(R\) is a circle of radius 2 in the \(xy\)-plane we again want to use polar coordinates. So we have

\[
\iint_S d\sigma = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1} \, r \, dr \, d\theta = \frac{\pi}{6} \left(17\sqrt{17} - 1\right)
\]
Example 3: Integrate the function $f(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ in the first octant.

The surface in the first octant looks like the following

![Diagram of the surface in the first octant](image)

The function describing the surface is $g(x, y, z) = 2x + 2y + z$

We have some freedom in choosing which coordinate plane to project the surface $S$ onto. Just to mix things up, let’s project $S$ into the $yz$-plane. Then the projection vector is $p = i$.

Then projection $R$ in the $yz$-plane looks like

![Diagram of the projection in the $yz$-plane](image)

Computing the Jacobian we have

$$\nabla g = \langle 2, 2, 1 \rangle \quad \Rightarrow \quad |\nabla g| = 3 \quad \text{and} \quad |\nabla g \cdot p| = |\nabla g \cdot i| = 2$$
Substituting into the surface integral we have

\[ \iint_S (x + y + z) \, d\sigma = \iint_R (x + y + z) \frac{3}{2} \, dA \]

Since we’ve projected \( S \) into the \( yz \)-plane we must eliminate \( x \) from the integrand. Solving the surface equation for \( x \) we have

\[ 2x + 2y + z = 2 \quad \Rightarrow \quad x = 1 - y - \frac{z}{2} \]

Substituting this into the integrand and setting up the limits of integration for \( R \) we have

\[ \iint_R (x + y + z) \frac{3}{2} \, dA = \frac{3}{2} \int_0^1 \int_0^{2-2y} \left(1 + \frac{z}{2}\right) \, dz \, dy = 2 \]

Sometimes we want to integrate over a surface \( S \) that is the union of multiple surfaces, say \( S = S_1 \cup S_2 \cup S_3 \). To integrate over \( S \) we break the integral up into multiple integrals over each of the constituent surfaces:

\[ \iint_S f \, d\sigma = \iint_{S_1} f \, d\sigma + \iint_{S_2} f \, d\sigma + \iint_{S_3} f \, d\sigma \]

**Example 4:** Integrate the function \( f(x, y, z) = xyz \) over the surface of the unit cube in the first octant.

The unit cube in the first octant looks like

The surface \( S \) of the cube is the union of its six faces. In theory we should compute six separate integrals, one for each face. However, we see that the function we’re integrating, \( f(x, y, z) = xyz \) is zero on the three sides of the cube lying in the coordinate planes, so we can skip them. We need to integrate over the top, side, and front facing sides which I’ve labeled \( A, B, \) and \( C \), respectively. We handle each one separately.
Side A: The top surface of the cube lies in the plane \( g(x, y, z) = z = 1 \). We must project the surface into the \( xy \)-plane since projecting into either of the other coordinate planes with result in a region \( R \) that is just a straight line. Note that the projection in the \( y \)-plane is the unit square in the first quadrant.

Computing the Jacobian of the transformation, we have

\[
\nabla g = \langle 0, 0, 1 \rangle \quad \Rightarrow \quad |\nabla g| = 1 \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{k}| = 1
\]

which gives \( d\sigma = dA \). This should not be surprising since the surface is a plane parallel to the \( xy \)-plane. We then have

\[
\iint_{S_A} (xyz) \, d\sigma = \iint_{R} (xyz) \, dA
\]

We use the surface equation \( z = 1 \) to eliminate \( z \) from the integrand. Then

\[
\iint_{S_A} (xyz) \, d\sigma = \iint_{R} (xyz) \, dA = \int_0^1 \int_0^1 xy \, dx \, dy = \frac{1}{4}
\]

The other two surfaces of interest are similar. We have

Side B: The side surface of the cube lies in the plane \( g(x, y, z) = y = 1 \). Our only option is to project into the \( xz \)-plane choosing \( \mathbf{p} = \mathbf{j} \). We then have

\[
\nabla g = \langle 0, 1, 0 \rangle \quad \Rightarrow \quad |\nabla g| = 1 \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{j}| = 1 \quad \Rightarrow \quad d\sigma = dA
\]

Then

\[
\iint_{S_B} (xyz) \, d\sigma = \iint_{R} (xyz) \, dA = \int_0^1 \int_0^1 xz \, dx \, dy = \frac{1}{4}
\]

and for the front side we have

Side C: The side surface of the cube lies in the plane \( g(x, y, z) = x = 1 \). Projecting the surface into the \( yz \)-plane and choosing \( \mathbf{p} = \mathbf{i} \) we have

\[
\nabla g = \langle 1, 0, 0 \rangle \quad \Rightarrow \quad |\nabla g| = 1 \quad \text{and} \quad |\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{i}| = 1 \quad \Rightarrow \quad d\sigma = dA
\]

Then

\[
\iint_{S_C} (xyz) \, d\sigma = \iint_{R} (xyz) \, dA = \int_0^1 \int_0^1 yz \, dx \, dy = \frac{1}{4}
\]
Finally, we add the contributions from each nonzero face to obtain

\[ \iint_S (xyz) \, d\sigma = \iint_{S_A} (xyz) \, d\sigma + \iint_{S_B} (xyz) \, d\sigma + \iint_{S_C} (xyz) \, d\sigma = \frac{3}{4} \]

**Flux Through a Surface**

We want to compute the flux of a vector field through some surface $S$. If you think of $F$ as a fluid velocity field, and the surface $S$ as a thin permeable membrane, then the flux through the surface $S$ is the rate at which fluid is flowing through the membrane.

But before we can compute flux, we need to talk about what kinds of surfaces we can work on. The first requirement is that the surface be smooth, or at least a union of smooth surfaces. For example, the cube in the previous example is not smooth, but it is a union of its six faces which are themselves smooth. The second requirement on the surface is that it is *orientable*.

**Definition:** A smooth surface $S$ is *orientable* or *2-sided* if it is possible to define an outward pointing unit normal vector $n$ that varies continuously with position.

In other words, an orientable surface is one where if you move an outward pointing normal vector all the way around the surface it will be pointing in the same direction when it returns to where it started. Most surfaces that we can picture in real life are orientable, e.g. spheres, planes, etc. An example of a surface that is not orientable is the *Mobius Strip*.

We’re now ready to talk about flux through a surface. We proceed in a similar fashion to the definition of flux through a curve for a planar flow. Suppose the fluid velocity field can be written as $F = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$. We divide the surface into infinitesimally small surface area chunks $d\sigma$ and compute the flux through each one. We then add up the flux through the little chunks (via a surface integral) to obtain the total flux through the surface.

Consider the following surface area element shown with a unit outward pointing normal vector $n$ and the fluid velocity field $F$ evaluated at some point on $d\sigma$.

The fluid flowing through the surface area element does so in a direction normal to the surface. To compute this flux we take the dot product of $F$ with the unit normal vector $n$ and multiply by the area of the surface area element. In other words

\[ \text{flux through } d\sigma = (F \cdot n) \, d\sigma \]
To compute the total flux through the surface we add up all the small surface area elements to obtain

\[
\text{Flux} = \int_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma.
\]

Note that the quantity \((\mathbf{F} \cdot \mathbf{n})\) is a scalar function and so this looks just like the surface integrals we described in the previous section.

Before we proceed, we should say what we mean by **outward** pointing unit normal vector \(\mathbf{n}\). Usually, if the surface is curved in some way, we choose the normal vector to point towards the outside of the curve. For example, given the two possible normal vectors on the surface of a sphere, we choose the one that points **away** from the sphere’s center.

We compute the unit normal vector to the surface as

\[
\mathbf{n} = \pm \frac{\nabla g}{|\nabla g|}
\]

where the sign is chosen so that the vector points outward from the surface. Then we have

\[
\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot \pm \frac{\nabla g}{|\nabla g|}
\]

Recalling the definition of the transformation between \(d\sigma\) and \(dA\), we have

\[
d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} \, dA
\]

Combining these in the surface integral we have

\[
\text{Flux} = \int_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \int_R \mathbf{F} \cdot \pm \frac{\nabla g}{|\nabla g|} \frac{|\nabla g|}{|\nabla g \cdot \mathbf{p}|} \, dA = \int_R \mathbf{F} \cdot \pm \frac{\nabla g}{|\nabla g \cdot \mathbf{p}|} \, dA
\]
**Example 5:** Find the flux of the fluid velocity field $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ through the surface $y^2 + z^2 = 1$ cut by the planes $z = 0$, $x = 0$, and $x = 1$.

Let’s draw the surface along with its projection into the $xy$-plane.

Let’s compute the integrand of the Flux integral. First we need to choose the sign on the normal vector so that it’s pointing outward from the surface. We eliminated $\mathbf{n}$ from the Flux integral, but $\nabla g$ points in the same direction, so we need to choose the sign on $\nabla g$ so that it points outward from the surface. We have

$$\nabla g = \langle 0, 2y, 2z \rangle$$

To make it easier to see the direction, we plug some point on the surface into $\nabla g$. Let’s pick the point $(0, 0, 1)$. Then

$$\nabla g(0, 0, 1) = \langle 0, 0, 2 \rangle$$

which points straight up and outward from the surface. So in the expression for the Flux integral we choose $+\nabla g$. We also have

$$|\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{k}| = |2z| = 2z$$

Note that we don’t need the absolute value sign because $z$ is greater than zero everywhere on the surface. Then

$$\frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle 0, yz, z^2 \rangle \cdot \frac{\langle 0, 2y, 2z \rangle}{2z} = \frac{2y^2z + 2z^3}{z} = y^2 + z^2$$

Setting up the integral we have

$$\text{Flux} = \iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \int_{-1}^1 \int_0^1 (y^2 + z^2) \, dx \, dy$$

The integral is still not computable in it’s current state because we’re integrating with respect to $x$ and $y$ but have a $z$ in the integrand. We need to use the equation of the surface to
eliminate the $z$ variable. Recalling that the equation of the surface is $y^2 + z^2 = 1$ we have

$$\text{Flux} = \iint_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \int_{-1}^{1} \int_{0}^{1} dx dy = 2$$

**Example 6**: Find the flux of the fluid velocity field $\mathbf{F} = \langle -2, 2y, z \rangle$ through the cylinder $y = e^x$ in the first octant cut by the planes $y = 2$ and $z = 1$.

Note that the surface $y = e^x$ does not have a $z$ in it, so the surface looks like the curve $y = e^x$ in the $xy$-plane and then propagated straight up in the $z$-direction. We have now plotted the surface and its projection into the $yz$-plane.

Note that we chose $\mathbf{p} = \mathbf{i}$ and projected into the $yz$-plane. We could have just as easily chosen $\mathbf{p} = \mathbf{j}$ and projected into the $xz$-plane. Since we projected into the $yz$-plane our final integrand should not have any $x$’s in it.

To get the function $g$ that describes the surface, we need to move all the variables in the surface equation to one side of the equation. We then have

$$g(x, y, z) = y - e^x \Rightarrow \nabla g = \langle -e^x, 1, 0 \rangle$$

We need to choose the sign on $\nabla g$ so that it points outward from the surface (or away from the $yz$-plane). We want the $\mathbf{i}$-component of $\nabla g$ to be pointing in the positive $x$-direction, which means we need to flip the sign on $\nabla g$. So we choose

$$-\nabla g = \langle e^x, -1, 0 \rangle$$

and so $|\nabla g \cdot \mathbf{p}| = |\nabla g \cdot \mathbf{i}| = |e^x| = e^x$

Then the integrand of the Flux integral is

$$\mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot \mathbf{p}|} = \langle -2, 2y, z \rangle \cdot \frac{\langle e^x, -1, 0 \rangle}{e^x} = \frac{-2e^x - 2y}{e^x} = -2 - \frac{2y}{e^x}$$
But on the surface we have $y = e^x$ and so

$$\mathbf{F} \cdot \frac{-\nabla g}{|\nabla g \cdot p|} = -2 - \frac{2y}{y} = -4$$

Plugging this into the Flux integral and setting up the limits of integration on $R$ we have

$$\text{Flux} = \int\int_S (\mathbf{F} \cdot \mathbf{n}) \, d\sigma = \int_0^1 \int_1^2 (-4) \, dydz = -4$$