On the front of your bluebook print (1) your name, (2) 3310/Final, (3) FALL 2011 and (4) a grading table with room for 6 problems and a total score. Show all work in your bluebook. Textbooks, class notes and calculators are not permitted, although you are allowed to use a one-page reminder sheet.

Do problems 1, 2 and 3. Then, choose two of the three problems 4-6 on page 2. Indicate which problem you are skipping by putting an X through that number on your grading sheet.

Please sign your bluebook under the Honor Code to indicate that you have neither given nor received unauthorized assistance on this exam.

1. (30 points) Consider the set of equations $-3x + 4y - 6z = -5/2$, $6x - 8y + 12z = c$, where $c$ is some real number.
   (a) Write the system in matrix form $Ax = b$.
   (b) Define the fundamental subspaces range and cokernel for an arbitrary matrix. Find these spaces for $A$.
   (c) Define the Fredholm compatibility conditions (Fredholm alternative) for a general system $Ax = b$. Find a value of $c$ that satisfies these conditions for the system in (a).
   (d) For this $c$ value, find the general solution to (a). Write the solution as $x = w + z$, where $w \in \text{corg}(A)$ and $z \in \text{ker}(A)$.
   (e) Of the solutions in (d), which has the smallest Euclidean norm?

2. (30 points) State the fundamental theorem of linear algebra, then, for each property given below, write down a matrix $F$ with that property or explain why no such matrix exists.
   (a) $\text{rng}(F) = \text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$, and $\text{corg}(F) = \text{span}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.
   (b) $F = F^{-1}$.
   (c) The vector $\begin{pmatrix} 10 \\ 6 \\ 90 \end{pmatrix}$ is in the kernel of $F$, $\begin{pmatrix} -6 \\ 10 \\ 0 \end{pmatrix}$ is in the corange of $F$, and $\det F = 1$.
   (d) $F$ is real and symmetric with eigenvalues $1 + i$ and $1 - i$.
   (e) $F$ has an eigenvalue 2 with multiplicity two, but only one eigenvector.

3. (30 points) Let $q(x, y, z) = x^2 + 2xy + 3y^2 + 4yz + 4z^2$ be a quadratic form.
   (a) Write $q = x^T K x$ for a matrix $K$.
   (b) Compute the $LU$ decomposition of $K$.
   (c) Define what it means to be a positive definite matrix. Is $q$ a positive definite quadratic form? Explain.
   (d) What do (a)-(c) allow you to conclude about the eigenvalues of $K$? (DO NOT COMPUTE the $\lambda$’s)
   (e) $\lambda_1 = 2$ is an eigenvalue of $K$. Find its associated eigenvector.
   (f) What is the sum of the other two eigenvalues, $\lambda_2 + \lambda_3$? (Hint: You DO NOT compute $\lambda_2$ and $\lambda_3$ separately.)
Do **two** of the three problems 4-6 below, indicate which problem you are skipping by putting an X through that number on your grading sheet.

4. (20 points) Let \( S = \text{span}\{(x - 1), (x - 1)^2\} \).
   
   (a) Define **vector subspace**.
   
   (b) Show that \( S \) is a vector subspace of \( P(2) \), the space of quadratic polynomials.
   
   (c) What is \( \text{dim}(S) \)?
   
   (d) Define the \( L^2 \) **inner product** on the interval \((0, 2)\).
   
   (e) Find the orthogonal complement \( S^\perp \) to \( S \) in \( P(2) \) using the inner product in (d).

5. (20 points) Let \( L[x] = \begin{pmatrix} x_1 + 3x_2 \\ -x_1 + x_2 \\ 2x_2 \end{pmatrix} \) be a linear transformation.
   
   (a) Which vector space is the domain of \( L \)? Which vector space is the co-domain of \( L \)?
   
   (b) Suppose \( \{v_i : i = 1, \ldots, n\} \) is a basis for the domain of \( L \). What is \( n \)? Why? (DO NOT FIND \( v_i \).)
   
   (c) If \( L[v_1] = \begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix} \), what is \( v_1 \)?
   
   (d) Find a basis for the range of \( L \).

6. (20 points) Let \( Q \) be the matrix
   
   \[
   Q = I - 2 \frac{vv^T}{v^Tv}
   \]
   
   where \( v \) is an \( n \)-dimensional vector. This type of matrix is called a Householder reflection matrix.
   
   (a) What is the size of the matrix \( v^Tv \)? Of the matrix \( vv^T \)?
   
   (b) Suppose the vector \( v = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \). Show that \( Q^TQ = I \).
   
   (c) For an arbitrary \( v \), show that \( Q = Q^T \).
   
   (d) Show that \( Q^TQ = I \) for any arbitrary \( v \).