**Theorem:** Let $\mathbb{X}$ be a normed linear space and let $\mathbb{Y}$ be a Banach space. Then $\mathcal{B}(\mathbb{X}, \mathbb{Y})$, the set of all bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ is Banach with respect to the operator norm.

**Proof:**

- Let $(T_n)$ be a Cauchy sequence in $\mathcal{B}(\mathbb{X}, \mathbb{Y})$. Then we can get $||T_n - T_m||$ as small as we want for sufficiently large $m$ and $n$.

- Since $||T_nx - T_mx|| = ||(T_n - T_m)x|| \leq ||T_n - T_m|| ||x||$ and since we can get $||T_n - T_m||$ as small as we want, we have that, for each $x \in \mathbb{X}$ $(T_nx)$ is a Cauchy sequence in $\mathbb{Y}$.

- $\mathbb{Y}$ Banach $\Rightarrow \lim_{n \to \infty} T_nx$ exists in $\mathbb{Y}$. Let’s call the limit $Tx$. That is, we define
  \[ Tx = \lim_{n \to \infty} T_nx. \]
  It is easy to show that $T$ is a linear operator.

- We now show that $T$ is bounded.
  Note that since
  \[ ||T_nx - T_mx|| \leq ||T_n - T_m|| ||x|| \]
  and $(T_n)$ is Cauchy, we have that
  \[ ||T_nx - T_mx|| \leq ||x|| \]
  for sufficiently large $m, n$.
  Letting $m \to \infty$, this gives
  \[ ||T_nx - Tx|| \leq ||x|| \]
  for sufficiently large $n$.
  Thus,
  \[ ||Tx|| \leq ||T_nx - Tx|| + ||T_nx|| \leq ||x|| + ||T_nx|| \]
  \[ \leq ||x|| + ||T_n|| ||x|| \leq (1 + ||T_n||)||x|| \]
  and so
  \[ ||T|| \leq 1 + ||T_n||. \]
  So, $T_n$ bounded $\Rightarrow$ $T$ bounded and therefore $T \in \mathcal{B}(\mathbb{X}, \mathbb{Y})$.

- It remains to show that $T_n \to T$ in the sense that $||T_n - T|| \to 0$.
  Let $\varepsilon > 0$. $(T_n)$ Cauchy $\Rightarrow \exists n \in \mathbb{N}$ such that $||T_n - T_m|| < \varepsilon$ for all $m, n \geq N$.
  So, we have
  \[ ||T_nx - T_mx|| \leq ||T_n - T_m|| ||x|| < \varepsilon ||x|| \]
  for all $m, n \geq N$ and for all $x \in \mathbb{X}$. 

Letting $m \to \infty$, this becomes
\[ ||T_n x - T x|| < \varepsilon ||x|| \]
for all $n \geq N$ and for all $x \in X$.

Therefore
\[ \sup_{||x||=1} ||T_n x - T x|| \leq \sup_{||x||=1} \varepsilon ||x|| = \varepsilon. \]

for all $n \geq N$.

The left-hand side is $||T_n - T||$, so we have
\[ ||T_n - T|| \leq \varepsilon \quad \forall n \geq N \]

and so we can conclude that $||T_n - T|| \to 0$, as desired.

That is, we took a Cauchy sequence in $B(X, Y)$ and showed that it converged to a point (operator) $T \in B(X, Y)$. \qed