Exam #3  
Lecturer: Manuel Ladser

INSTRUCTIONS: On the front of your bluebook please print your name, student ID, date and course code. Show all your work in your bluebook. Please start each new problem on a new page. Solve the problems in the same order as they are requested. A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.

1. (40 points.) Consider a time-homogenous and irreducible Markov process \( X = (X_t)_{t \geq 0} \) over the space \( S = \{1, 2, 3, 4, 5\} \). As usual, \( Q \) denotes the rate matrix and \( \pi \) the stationary distribution of the process. Furthermore, \( p_t \) denotes its probability transition matrix.

Determine whether each of the following statements is TRUE (i.e. always true) or FALSE (i.e. not always true). Do not justify your answers!

(a) For all \( t \geq 0 \), \( \pi \cdot p_t = \pi \).

(b) \( Q \cdot p_t = 0 \), for each \( t > 0 \).

(c) For all \( i \in S \), \( \sum_{j \neq i} \pi(j) \cdot Q(j, i) = -\pi(i) \cdot Q(i, i) \).

(d) If \( h > 0 \) is a constant then the sequence of states \( X_0, X_h, X_{2h}, X_{3h}, \ldots \) is a first-order homogeneous Markov chain with probability transition matrix \( p_h \).

(e) For all \( s > 0 \), \( \lim_{t \to \infty} \mathbb{P}(X_{t+s+\sqrt{t}} = 1, X_{t+s} = 2 \mid X_s = 3) = \pi(1) \cdot p_s(3, 2) \).

(f) \( \lim_{t \to \infty} \frac{1}{t} \int_0^t \cos(X_t) \, dt = \sum_{i=1}^5 \sin(i) \cdot \pi(i) \).

(g) If the initial distribution of \( X \) satisfies the detailed-balance condition then, for all \( t > 0 \):

\[ \mathbb{P}(X_t = 4, X_{2t+1} = 1, X_{3t+2} = 5) = \mathbb{P}(X_0 = 5, X_{4/3} = 1, X_{5/3} = 4) \]

(h) If \( X \) is stationary then, for each \( T > 0 \), the reversed process \( (X_{T-t})_{0 \leq t \leq T} \) has probability transitions given by the expression \( r_t(i, j) = \pi(j) \cdot p_t(j, i) / \pi(i) \), for each \( i, j \in S \).

2. (30 points.) Two salesmen work in a small office selling insurance policies. Each salesman is either on the phone or off the phone, and calls to a busy salesman are lost. Furthermore, the \( i \)-th salesman receives phone calls at rate \( \lambda_i > 0 \) (when off the phone) and services each call (that makes it through) for an exponential amount of time with rate \( \mu_i > 0 \).

Using the state space \( \{0, 1, 2, 12\} \), where states represent the salesmen that are busy on the phone, respond:

(a) What's the rate matrix \( Q \) of the Markov process associated with the above state space?

(b) Show that up to a constant \( c \) the stationary distribution of the process is of the form \( \pi = c \cdot (\lambda_1 \lambda_2, \lambda_1 \lambda_3, \lambda_2 \lambda_1, \lambda_1 \lambda_2) \). Determine \( c \) explicitly!

(c) What fraction of the time is the first salesman off the phone? Do not justify your answer!

(d) At what rate are calls missed by the small office? Do not justify your answer!

(ONE MORE PROBLEM ON THE BACK!)
3. (30 points.) Consider a network of two M/M/1 queues in series with service rates $\mu_1$ and $\mu_2$, respectively (see diagram below). New customers arrive at the first queue with a certain rate $\lambda$ and, after being served, either enter the second queue (with probability $\alpha < 1$) or exit the system. Instead, after being served, customers in the second queue go back to the first queue (with probability $\beta < 1$) or exit the system entirely.

Based on the above description, respond:

(a) Determine the transition rates out of a generic state $(m, n)$, with $m, n > 0$, of the Markov process associated with the sizes of each queue.

(b) Let $r_1$ and $r_2$ denote the output rates from the first and second queue, respectively, when the network is equilibrium (assuming this is possible). Determine a linear system satisfied by $r_1$ and $r_2$, and solve it explicitly.

(c) Under what conditions can the network achieve equilibrium? In this case, what is its stationary distribution $\pi$? Do not justify your answers!
\( P1 \)

(a) **TRUE b/c**
\[
\Pi \cdot \Pi_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Pi \cdot a_k = \Pi \text{ since } \Pi \cdot Q = 0
\]

(b) **FALSE b/c**
if \( Q \cdot \Pi_t = 0 \) then \( \Pi_t = 0 \) i.e. \( \Pi_t = I \)
for all \( t \geq 0 \), which means not to be the case.

(c) **TRUE b/c**
\( \Pi \cdot Q = 0 \) means that for each \( i \):
\[
(\Pi \cdot Q)(i) = \sum_{j \neq i} 1 \text{ if } j \in Q(i)
\]

(d) **TRUE b/c**
\[
P(X_{k+1} = i_k | X_{k-1} = i_{k-1}, \ldots) \iff P(X_{k+1} = i_k | X_{k-1} = i_{k-1})
\]

(e) **FALSE b/c**
\[
\lim_{t \to \infty} P(X_{t+5+\pi} = 1, X_{t+5} = 2 | X_5 = 3)
\]
\[
= \lim_{t \to \infty} P(X_{t+5+\pi} = 1 | X_{t+5} = 2) \cdot P(X_{t+5} = 2 | X_5 = 3)
\]
\[
= \lim_{t \to \infty} P(s(1) \cdot P_t(2, 1) = \Pi(3, 2) = \Pi(3, 2) = \Pi(3, 2) = \Pi(3, 2) = \Pi(3, 2) = \Pi(3, 2)
\]

(f) **FALSE b/c**
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \cos(X_t) \, dt = \frac{5}{2} \cos(i) \cdot \Pi(i)
\]

(g) **FALSE b/c**
Even though \( X \) is time reversible:

\[
\begin{array}{cccc}
\text{t} & \text{2t+1} & \text{3t+1} \\
\downarrow & \downarrow & \downarrow \\
0 & \frac{1}{2} & \frac{5}{2} \\
& \frac{1}{2} & \frac{1}{2} \\
& \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{array}
\]

and \( \frac{1}{2} \neq \frac{1}{3} \)
TRUE by seeing in lecture.

\[ Q = \begin{bmatrix}
0 & -\mu_1 & \mu_2 & 0 \\
-\mu_1 & 0 & 0 & 0 \\
-\mu_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ \pi \cdot Q = 0 \]

\( \pi_0 (o) = c \cdot \frac{1}{\mu_1 \mu_2} (\mu_1 + \mu_2) + \frac{\mu_1}{\mu_1 \mu_2} + \frac{\mu_2}{\mu_1 \mu_2} \]

\( \pi_1 (1) = c \cdot \frac{1}{\mu_1 \mu_2} - \frac{\mu_1}{\mu_1 \mu_2} (\mu_1 + \mu_2) + \frac{\mu_2}{\mu_1 \mu_2} \]

\( \pi_2 (2) = c \cdot \frac{1}{\mu_1 \mu_2} - \frac{\mu_2}{\mu_1 \mu_2} \mu_1 (\mu_2 + \mu_1) + \frac{\mu_1}{\mu_1 \mu_2} \]

\( \pi_2 (12) = c \cdot \frac{1}{\mu_1 \mu_2} \mu_1 \mu_2 + \frac{\mu_2}{\mu_1 \mu_2} - \frac{\mu_1}{\mu_1 \mu_2} (\mu_1 + \mu_2) \]

i.e. \( \pi \cdot Q = 0 \)

To have \( \pi \) a state distr. Therefore we just need

\[ c = \frac{1}{\mu_1 \mu_2 + \mu_1 \mu_2 + \mu_1 \mu_2 + \mu_1 \mu_2} \]

\[ \text{ANS} = \pi (1) + \pi (12) \]

\[ \text{ANS} = \pi_1 (1) + \pi_2 (12) + (\pi_1 + \pi_2) \pi (12) \]

\[ (m-1, m+1) \]

\[ (m-1, m) \leftarrow (m, m) \rightarrow (m+1, m) \]

\[ (m, m-1) \rightarrow (m+1, m-1) \]
1. $\gamma \cdot n_2 \cdot \beta = n_1$
   
   In particular: $\gamma + \alpha \beta \cdot n_1 = n_1$, i.e.

   $n_1 = \frac{\alpha \beta}{1-\alpha \beta}$

   and

   $n_2 = \frac{\alpha \gamma}{1-\alpha \beta}$

2. The network will achieve equilibrium when $\frac{n_1}{\mu_1}$ and $\frac{n_2}{\mu_2}$. In this case, by a result discussed in lecture:

   \[
   \pi(m, n) = \left(1 - \frac{n_1}{m1}\right) \left(\frac{n_1}{m1}\right)^m \cdot \left(1 - \frac{n_2}{m2}\right) \left(\frac{n_2}{m2}\right)^n
   \]