Path Connectedness

Definition: A topological space $\mathbb{X}$ is **path connected** if for any two points $x, y \in \mathbb{X}$ there exists a continuous $f : [0,1] \rightarrow \mathbb{X}$ such that $f(0) = x$ and $f(1) = y$.

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**Theorem:** If $\mathbb{X}$ is path connected then $\mathbb{X}$ is connected.

**Proof:**

- Let $\mathbb{X}$ be path connected. Suppose that it is not connected.
- Then there exist disjoint, non-empty, open $U, V \subseteq \mathbb{X}$ such that $\mathbb{X} = U \cup V$.
- Choose points $x \in U$ and $y \in V$. By the assumption that $\mathbb{X}$ is path connected, there exists a continuous $f : [0,1] \rightarrow \mathbb{X}$ such that $f(0) = x$ and $f(1) = y$.
- Consider $f^{-1}(U)$ and $f^{-1}(V)$. Note the following facts about these sets.
  - They are disjoint in $[0,1]$ and their union is $[0,1]$.
  - By continuity of $f$, they are both open in $[0,1]$.
  - Since $0 \in f^{-1}(U)$ and $1 \in f^{-1}(V)$, they are non-empty.
- So, $f^{-1}(U)$ and $f^{-1}(V)$ partition $[0,1]$ into disjoint, non-empty, open sets. However, this contradicts the fact that $[0,1]$, as an interval of $\mathbb{R}$, is connected.
  Thus, $\mathbb{X}$ must be connected.

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**Note:** The converse is not true. $\mathbb{X}$ may be connected but not path connected. As a famous example of this, please look up the “topologist’s sine curve”.