Exam Review Chapters $5, 6, 7$

1. (Fall 2011 Final #7) Consider the matrix:

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

   (a) Find all the eigenvalues of A.

   (b) For each eigenvalue that you found in part (a), find a basis for the corresponding eigenspace.

   (c) Find the general solution of $\vec{x}' = A\vec{x}$.

   solution:

   (a) $p(\lambda) = |A - \lambda I| = (2-\lambda)[(1-\lambda)^2 - 1] = (2-\lambda)(\lambda-2)\lambda$. Setting $p(\lambda) = 0$ yields $\lambda_1 = 0,\ \lambda_{2,3} = 2$.

   (b) To find a basis for each eigenspace, we find the eigenvectors associated with each eigenvalue.

Eigenvalue	Eigenvector
$\lambda = 0$	$\mathbf{v_1} = [0, 1, -1]^T$
$\lambda = 2$	$\mathbf{v_2} = [0, 1, 1]^T$

   Then, the basis for $\mathbb{E}_{\lambda=0}$ is $\mathbf{v_1}$ and the basis for $\mathbb{E}_{\lambda=2}$ is $\mathbf{v_2}$.

   (c) To find the general solution, we must construct a generalized eigenvector for $\lambda = 2$. We seek a vector $\mathbf{u}$ such that $(A - 2I)\mathbf{u} = \mathbf{v_2}$. We find $\mathbf{u} = [2, 1, 0]^T$. The general solution is:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{2t} \left(t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right)$$

2. (Fall 2011 Final #8) Consider the following system of nonlinear equations:

$$\left\{ \begin{array}{ccc} \frac{dx}{dt} & = & y(x-1) \\ \frac{dy}{dt} & = & (x-y)(x+y-1) \end{array} \right\}$$

   (a) Find all the equilibrium points.

   (b) Classify the stability of each equilibrium point.

   solution:

(a) To find the equilibrium points, we set $x' = 0$ and $y' = 0$ simultaneously. When $x' = 0$, we have $y = 0$ and $x = 1$. When $y' = 0$, we have $x = y$ and $y = 1 - x$. Therefore, when both $x' = 0$ and $y' = 0$, we find equilibrium points at $(0,0)$, $(1,0)$, $(1,1)$.

(b) To calculate the stability of each equilibrium point, we linearize the system and analyze the eigenvalues of the Jacobian matrix at each equilibrium point.

$$J = \begin{bmatrix} y & x-1 \\ 2x-1 & 1-2y \end{bmatrix}$$

Evaluating the matrix at each of the equilibrium points yields the following:

Equilibrium Point	Eigenvalues	Stability
$(0,0)$	$\lambda_{1,2} = \frac{1\pm\sqrt{5}}{2}$	unstable
$(1,0)$	$\lambda_1 = 1,\ \lambda_2 = 0$	unstable
$(1,1)$	$\lambda_1 = 1,\ \lambda_2 = -1$	unstable

3. (Fall 2010 Final #2) Consider the system

$$\dot{x} = 20 - x^2 - y^2$$
$$\dot{y} = 8 - xy$$

(a) Find the equilibrium points for this system.

(b) Classify the stability of each equilibrium point.

(c) Sketch the phase plane, showing all equilibria, nullclines, and some possible solution curves.

Solution:

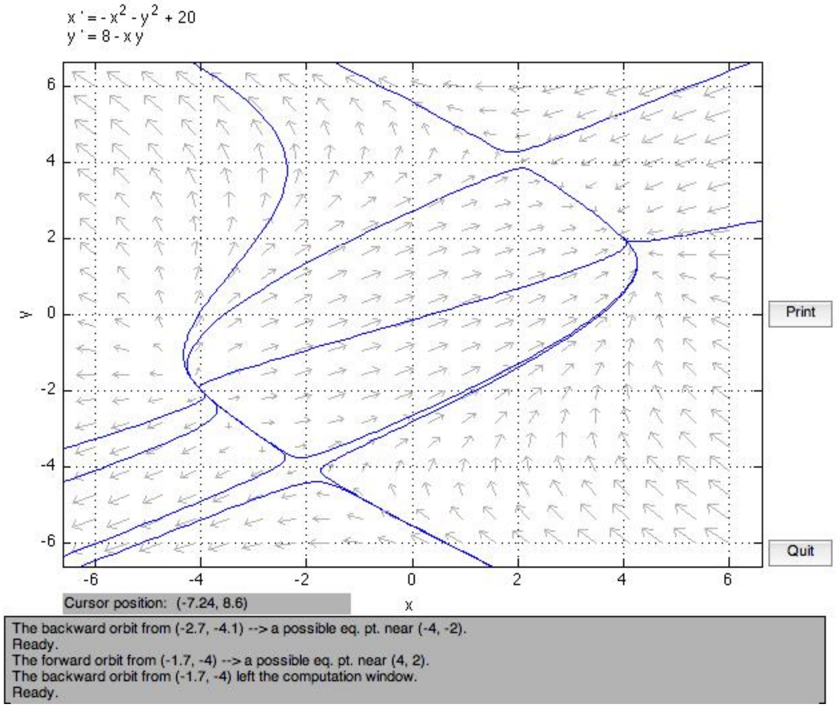
(a) The equilibrium points are found when $x' = 0$ and $y' = 0$ simultaneously. Setting $x' = 0$, we have $x^2 + y^2 = 20$. Setting $y' = 0$, we have $y = \frac{8}{x}$. We solve these equations simultaneously by plugging $y = \frac{8}{x}$ into $x^2 + y^2 = 20$. Doing this, we find four equilibrium points: $(4,2)$, $(-4,-2)$, $(2,4)$, $(-2,-4)$.

(b) We find the stability of each equilibrium point by analyzing the eigenvalues of the Jacobian matrix at each equilibrium point.

$$J = \begin{bmatrix} -2x & -2y \\ -y & -x \end{bmatrix}$$

Evaluating the matrix at each of the equilibrium points yields the following:

Equilibrium Point	Eigenvalues	Stability
$(4,2)$	$\lambda_{1,2} = -6 \pm \sqrt{10}$	stable
$(-4,-2)$	$\lambda_{1,2} = 6 \pm \sqrt{10}$	unstable
$(2,4)$	$\lambda_{1,2} = -3 \pm \frac{2\sqrt{33}}{3}$	unstable
$(-2,-4)$	$\lambda_{1,2} = -3 \pm \frac{2\sqrt{33}}{3}$	unstable



(c)

4. (Fall 2010 Final and assorted) Answer the following TRUE or FALSE.

   (a) Let $\lambda_1,\ \lambda_2,\ \lambda_3$ be the eigenvalues of

   $$A = \begin{bmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{bmatrix}$$

   Then $\lambda_1\lambda_2\lambda_3 = 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = (a + e)$.

   (b) The second order ODE $y'' + \omega^2 y = G(y)$ is integrable (where $G(y)$ is a nonlinear function of y).

   (c) Let $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ be a 2×2 nonlinear system. Let $(x_e,\ y_e)$ be an equilibrium solution to this system with a linearization that results in the eigenvalues $\lambda_1 = 2i,\ \lambda_2 = -2i$. It follows that the behavior of the nonlinear system in the local neighborhood of the equilibrium point is a center.

   (d) Given the system $\vec{x}' = A\vec{x}$, answer each of the following either TRUE or FALSE

      i. $\vec{x} = \vec{0}$ is the only equilibrium point.

      ii. The point $\vec{0}$ is a sink if A has a negative (real) eigenvalue.

      iii. A has n eigenvalues and n linearly independent eigenvectors.

      iv. Assume A is a 3×3 matrix. If A has one eigenvalue $\lambda_1 = 2$ and a double eigenvalue $\lambda_{2,3} = 1$, then the general solution contains a generalized eigenvector if the geometric multiplicity of $\lambda_{2,3}$ is one.

   (e) A limit cycle is attracting if the eigenvalue of the limit cycle is less than zero.

<u>solution:</u>

(a) TRUE

(b) TRUE

(c) FALSE

(d) i. FALSE

ii. FALSE

iii. FALSE

iv. TRUE

(e) FALSE

5. (Spring 2007, Exam 3) Consider the second order, linear, homogeneous ODE

$$x'' - x' - 6x = 0.$$

(a) Write the above equation in the matrix-vector form $\vec{w}' = A\vec{w}$.

(b) Find the general solution of the system derived in (a).

(c) Sketch the phase portrait of the system derived in (a).

(d) Find and classify the equilibria for the system derived in (a), and determine the behavior of solutions as $t \to \infty$.

<u>solution:</u>

(a)

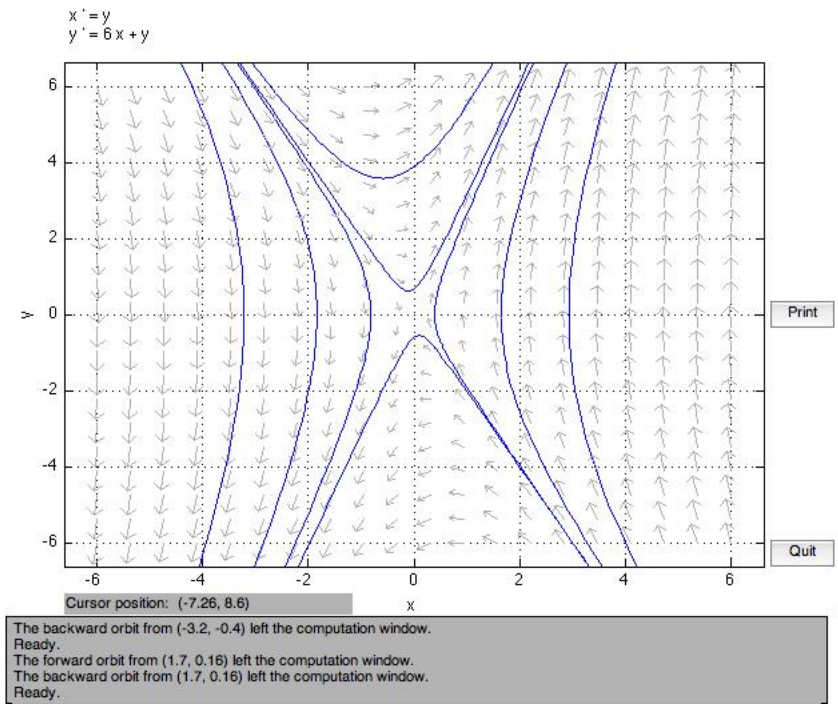
$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

(b) We find the eigenvalues and associated eigenvectors to be:

$$\lambda_1 = 3, \mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}; \lambda_2 = -2, \mathbf{v_2} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Our solution is

$$\mathbf{w}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(c) 

(d) The equilibrium point is $(0, 0)$ and it is a saddle (seen by the eigenvalues of opposite sign). As $t \to \infty$, solutions will go to infinity. They will approach (asymptotically) trajectories parallel to the eigenvector $\mathbf{v_1}$.