Basis worksheet

1.

(a) 7×4 matrix with linearly independent columns

$\Rightarrow$ every column is a pivot column

$\Rightarrow \text{rank}(A) = 4$

$\Rightarrow$ 3 rows will be all zeros

(b) Let $D = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$. Then, $D^TD = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix} \Rightarrow \text{rank}(D^TD) = 4$

(c) YES! $\det(D^TD) = 1 \neq 0$

(d) $DD^T = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \end{bmatrix} \Rightarrow \text{rank}(DD^T) = 4$

(e)

NO! $\det(DD^T) = 0$.

2. $\{\sin(at), \cos(at+a), 1\}$

This set is guaranteed to be linearly independent if the Wronskian is non-zero.

$$W = \begin{vmatrix} \sin(at) & \cos(at+a) & 1 \\ a\cos(at) & -a\sin(at+a) & 0 \\ -a^2\sin(at) & -a^2\cos(at+a) & 0 \end{vmatrix}$$

$$W = 1 \cdot \begin{vmatrix} a\cos(at) & -a\sin(at+a) \\ -a^2\sin(at) & -a^2\cos(at+a) \end{vmatrix}$$

$$= -a^3\cos(at)\cos(at+a) - a^3\sin(at)\sin(at+a)$$
$$= -a^3[\cos(at)\cos(at+a) + \sin(at)\sin(at+a)]$$
$$= -a^3\cos(at-(at+a))$$
$$= -a^3 \cdot \cos(-a) = -a^3 \cdot \cos(a)$$

$\Rightarrow W=0$ if and only if $a=0$ or $\cos(a)=0$.

When $a=0$, the set is $\{0, 1, 1\}$ which is linearly dependent!

When $\cos(a)=0$, $a = \frac{\pi}{2}+k\pi$,

Use $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

$\cos\left(\left(\frac{\pi}{2}+k\pi\right)t + \left(\frac{\pi}{2}+k\pi\right)\right) = \pm\sin\left(\left(\frac{\pi}{2}+k\pi\right)t\right) = \pm\sin(at)$, so the set is linearly dependent.

$\boxed{a \neq 0 \text{ and } a \neq \frac{\pi}{2}+k\pi,\ k=0,\pm1,\pm2,\ldots}$

③ $2x+y-z+w=0$

$y+z=0$

$$\begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \overset{R_1'=\frac{1}{2}R_1}{\sim} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\overset{R_1'=-\frac{1}{2}R_2+R_1}{\sim} \begin{bmatrix} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$x-z+\frac{1}{2}w=0$ $\quad$ z free

$y+z \quad =0$ $\quad$ w free

$\Rightarrow$ a basis of V is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

let $z=1, w=0$ $\quad$ let $z=0, w=1$

$\boxed{\dim(V)=2.}$

④ $$\begin{bmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & \alpha \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{\alpha} & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & \alpha \end{bmatrix}$$ when $\alpha \neq 0$, if $\alpha=0$ then $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ clearly has 0 determinant

$$\sim \begin{bmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & \alpha-\frac{1}{\alpha} & 1 \\ 0 & 1 & \alpha \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & 1 & \alpha \\ 0 & \frac{\alpha^2-1}{\alpha} & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 2-\alpha^2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{\alpha} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

true only if $2-\alpha^2 \neq 0 \rightarrow \alpha \neq \pm\sqrt{2}$

$\boxed{\text{Therefore, the vectors are linearly dependent if } \alpha=0, \pm\sqrt{2}.}$

Alternatively, we could find the determinant of $A = \begin{bmatrix} \alpha & 1 & 0 \\ 1 & \alpha & 1 \\ 0 & 1 & \alpha \end{bmatrix}$ and solve for when this is zero.

$$\begin{aligned} \det A &= \alpha(\alpha^2-1) - 1(\alpha) \\ &= \alpha^3 - 2\alpha \\ &= \alpha(\alpha^2-2) \\ &= 0 \end{aligned}$$

$$\boxed{\Rightarrow \alpha = 0 \text{ or } \alpha = \pm\sqrt{2}}$$

(5.)

$$\begin{aligned} x+y+z&=0 \\ y-z&=0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$x+2z=0$
$y-z=0$
z free

$\Rightarrow$ a basis of W is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

let $z=1$

$$\boxed{\dim(W) = 1}$$

(6.)

If $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is not invertible, calculate A^n, $n>1$.

Since A is not invertible, $\det(A) = 0$.

$\det(A) = -a^2 - bc$
$\Rightarrow a^2 = -bc$

$$A^2 = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = \begin{bmatrix} a^2+bc & 0 \\ 0 & a^2+bc \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ for } n>1.$$

(7.) (a) $\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $\underline{u_1}$ and $\underline{u_2}$ are linearly independent

(b) We need to find $\underline{u_3}$ such that $c_1\underline{u_1} + c_2\underline{u_2} \neq \underline{u_3}$ for all $c_1, c_2 \in \mathbb{R}$.

All linear combinations of $\underline{u_1}, \underline{u_2}$ satisfy $c_1\underline{u_1} + c_2\underline{u_2} = \begin{bmatrix} c_1 \\ 0 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ -c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_2 \\ c_1 \end{bmatrix}$

Thus, we can choose any $\underline{u_3}$ not equal to $\begin{bmatrix} c_1 + c_2 \\ -c_2 \\ c_1 \end{bmatrix}$ for arbitrary c_1 and c_2.

One example is $\underline{u_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ obtained by setting $c_1 = c_2 = 1$.

(c) a basis for $\mathbb{R}^3$ is $\{\underline{u_1}, \underline{u_2}, \underline{u_3}\}$

(8.)

Let $S = \{t^2, t+3, t^3+4, t-1, t^2-5t+1\}$.

Is S a spanning set for $\mathbb{P}_3$?

We need to show that $c_1 t^2 + c_2(t+3) + c_3(t^3+4) + c_4(t-1) + c_5(t^2-5t+1)$
$= a_0 + a_1 t + a_2 t^2 + a_3 t^3$ for any $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

$$\underbrace{\begin{bmatrix} 0 & 3 & 4 & -1 & 1 \\ 0 & 1 & 0 & 1 & -5 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\underline{b}}$$

This has a solution (an infinite number) if it is consistent.

We need to show that A has four linearly independent columns. Then, it will be consistent.

We can row-reduce A to

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}.$$

Therefore, S <u>is</u> a spanning set for $\mathbb{P}_3$.

(9.)

$S = \{ [a, a-b, 2a+3b] \mid a, b \in \mathbb{R} \}$

A basis for S is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \right\}$ obtained by setting $a=1, b=0$, then $a=0, b=1$

$\dim(S) = 2$

(10.)

vector space

subspace

subset

basis

spanning set

linear dependence