Vector space/subspace worksheet

① $[A|b]$ $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & -1 & 0 & 0 \\ -1 & 1 & 1 & \lambda \end{array}\right] \underset{R_3^* = R_3 + R_1}{\overset{R_2^* = R_2 - 2R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & -2 & -4 \\ 0 & 1 & 2 & \lambda+2 \end{array}\right] \overset{R_3^* = R_2 + R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & 0 & \lambda-2 \end{array}\right]$

The system is consistent for $\lambda = 2$.

There are an infinite number of solutions.

②

ⓐ $p_1(x) = a_0 + a_1 x^2 + \ldots + a_n x^{2n}$

$p_2(x) = b_0 + b_1 x^2 + \ldots + b_n x^{2n}$

$p_1 + p_2 = (a_0 + b_0) + (a_1 + b_1) x^2 + \ldots + (a_n + b_n) x^{2n}$

$cp_1 = (ca_0) + (ca_1) x^2 + \ldots + (ca_n) x^{2n}$

$\boxed{\text{YES}}$

ⓑ $y_1'' + y_1 = 2$

$y_2'' + y_2 = 2$

$$(y_1 + y_2)'' + (y_1 + y_2) = (y_1'' + y_1) + (y_2'' + y_2)$$
$$= 2 + 2$$
$$= 4$$
$$\neq 2$$

$\boxed{\text{NO}}$

ⓒ Let A be a 7×7 matrix with $\det A = 1$.

Then,

$$|cA| = c^7 |A| \neq c|A|$$

$\boxed{\text{NO}}$

(3.)

$\det C^2 = \det C \cdot \det C$
$\det C^3 = \det C^2 \cdot \det C$

$$\Rightarrow \begin{cases} 2 = \det C \cdot \det C & \Rightarrow \det C = \sqrt{2} \\ 3 = \det C^2 \cdot \det C & \Rightarrow \det C = \frac{3}{2} \end{cases}$$

$\boxed{\text{NO}}$

(4.)

(a) $(A+B)^{-1}(A+B) = I = (A+B)(A+B)^{-1}$

Does $(A+B)^{-1} = B^{-1} + A^{-1}$?

$$\begin{aligned} I = (A+B)^{-1}(A+B) &\overset{?}{=} (B^{-1}+A^{-1})(A+B) \\ &= B^{-1}A + B^{-1}B + A^{-1}A + A^{-1}B \\ &= B^{-1}A + 2I + A^{-1}B \\ &\neq I \end{aligned}$$

$\boxed{\text{FALSE}}$

(b) Since $|X| \neq 0$, X^{-1} exists.

$AX = XB$

$\Rightarrow AXX^{-1} = XBX^{-1}$

$\Rightarrow A = XBX^{-1} \neq B$

$\boxed{\text{FALSE}}$

(c) Consider
$$\begin{aligned} a_1x_1 + a_2x_2 &= d_1 \\ b_1x_1 + b_2x_2 &= d_2 \\ c_1x_1 + c_2x_2 &= d_3. \end{aligned}$$

$$\left[\begin{array}{cc|c} a_1 & a_2 & d_1 \\ b_1 & b_2 & d_2 \\ c_1 & c_2 & d_3 \end{array}\right] \text{ could perhaps be row-reduced to } \left[\begin{array}{cc|c} 1 & 0 & r_1 \\ 0 & 1 & r_2 \\ 0 & 0 & 0 \end{array}\right]$$

in which case the answer is $\boxed{\text{TRUE}}$

(d)

$$A = B^{-1}CB \Rightarrow |A| = |B^{-1}CB| = |B^{-1}|\,|CB|$$
$$= |B^{-1}|\,|C|\,|B|$$
$$= |C|\,|B^{-1}|\,|B|$$
$$= |C|\,|B^{-1}B|$$
$$= |C|$$

$\boxed{\text{TRUE}}$

5.

$$F(x) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

(a) $$F(x)\cdot F(y) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x+y & 1 \end{bmatrix} = F(x+y)$$ ✓

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ x & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_2^* = R_2 - xR_1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -x & 1 \end{array}\right]$$

$$\Rightarrow F^{-1} = \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix}$$

(b) $$\frac{dF}{dx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \frac{dF^{-1}}{dx} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\frac{dF}{dx} + \frac{dF^{-1}}{dx} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{0}$$ ✓

(c)

$$e^{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \frac{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}{2} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$