Final Exam Review: Chapter 3

1. Consider the following system,

$$\begin{aligned} x_1 + 4x_2 - 5x_3 &= 0 \\ 2x_1 - x_2 + 8x_3 &= 9 \\ x_1 - 5x_2 + 13x_3 &= 9. \end{aligned}$$

   (a) The system can be written as a matrix equation $A\vec{x} = \vec{b}$. What are A, $\vec{x}$ and $\vec{b}$?

   (b) Reduce the associated augmented matrix to reduced row-echelon form.

   (c) Find $\vec{x_h}$, the solution of the homogeneous equation $A\vec{x} = \vec{0}$.

   (d) Find $\vec{x_p}$, a particular solution of $A\vec{x} = \vec{b}$.

   (e) Use the nonhomogeneous principle to write down the general solution of the system of equations.

2. Determine whether or not each of the following is a vector space. If it is a vector space, prove it. If not, show why not. **Hint:** Use the subspace theorem. You may assume that $\mathbb{R}^n, \mathbb{P}_n, \mathbb{M}_{mn}$ and $C(I)$ are all vector spaces.

   (a) $V =$ the set of all points in $\mathbb{R}^2$ that are on the line $y = x$.

   (b) $V =$ the set of all invertible 2×2 matrices.

   (c) $V =$ the set of continuous functions, f on $[0, 1]$ such that $f(1) = 0$.

   (d) $V = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$

   (e) $V =$ the set of *symmetric* $n \times n$ matrices, $\{A \in \mathbb{M}_{nn} : A^T = A\}$.

3. For a given system of equations

$$\begin{aligned} \alpha x + y - z &= 1 \\ y + z &= 3 \\ x - 2y &= \beta, \end{aligned}$$

   which values of α and β does the system

   (a) have a unique solution?

   (b) have infinitely many solutions?

   (c) have no solutions?

(Please state explicitly and clearly for both α and β.)

4. If the matrix $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is *not* invertible, calculate $A^n, n > 1$.

5. True/False. If the statement is *always* true, mark true. Otherwise, mark false. You do not need to show your work. (Any work will not be graded.)

   (a) If $x(t)$ is the solution to the equation of an overdamped spring, then there does not exist a finite value t_1 such that $x(t_1) = 0$.

   (b) If a set of vectors $W = \{w_1, w_2, \ldots, w_n\}$ is linearly independent in a vector space V, then any subset of W is also linearly independent in V.

   (c) Let A and B be invertible $n \times n$ matrices. Then $|(AB)^{-1}| = \frac{1}{|A|\cdot|B|}$.

   (d) If $\vec{x} = \vec{0}$ is a solution to $A\vec{x} = \vec{0}$, then $|A| \neq 0$.

   (e) Let $\{v_1, \ldots, v_n\}$ be a basis for a vector space V. Then for any $y \in V$, $\{v_1, \ldots, v_n, y\}$ is linearly dependent.

   (f) There exists a real 3×3 matrix, A, with eigenvalues $\lambda_1 = 1, \lambda_2 = i, \lambda_3 = 3$.

   (g) If A, B and $A + B$ are all invertible matrices, then $(A + B)^{-1} = B^{-1} + A^{-1}$.

6. Let
$$A = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$$

   (a) Find A^{-1} or explain why it does not exist.

   (b) Use your answer to (a) to find the solution(s) of $A\vec{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ or explain why no solutions exist.

   (c) Are there any vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution? If so, what are the conditions on $\vec{b}$; if not, why not?