Synched: Model Descriptions

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Synched includes four built-in models to study synchronizing phenomena:

1. The **Kuramoto** model,
2. a **Time-Delayed** Kuramoto model,
3. a **Cluster** synchrony model, and
4. a Kuramoto model on a network with **Community** structure.

Below the governing equations and order parameters for each model are described. Interpretations for each parameter and order parameter are also given. For detailed analysis of all four models, including low-dimensional descriptions and bifurcation diagrams, see References [1–4] given below. After each model is introduced, forcing is discussed.

I. KURAMOTO

The **Kuramoto** model [1] is a well-studied system that has become a paradigm for modeling synchronizing phenomena, and the simplest model included in **Synched**. It consists of N oscillators, each described by their phase θ_n, that evolve according to the ODEs

$$\dot{\theta}_n = \omega_n + \frac{K}{N}\sum_{m=1}^{N}\sin(\theta_m - \theta_n), \tag{1}$$

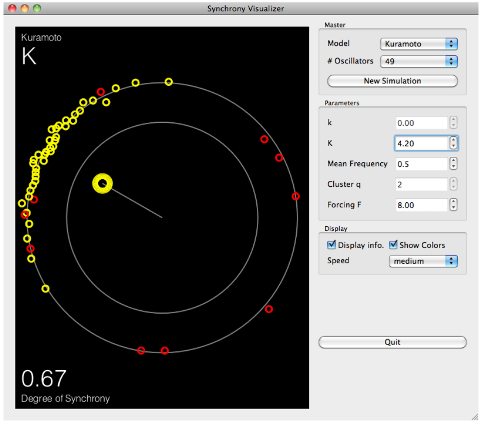
where ω_n is the *intrinsic frequency* of oscillator n and K is the global coupling strength. Note that coupling is *all-to-all*, i.e. every oscillator is coupled to every other oscillator. In **Synched**, each ω_n is drawn from the distribution

$$g(\omega) = \frac{\pi^{-1}}{1 + (\omega - \Omega)^2}. \tag{2}$$

Thus, the two relevant parameters are K and Ω. In **Synched**, the global coupling strength K can be modulated between 0 and 20, and the mean frequency Ω can be modulated between -10 and 10.

To quantify synchrony in the system, we use the standard order parameter

$$R = \frac{1}{N}\sum_{m=1}^{N} e^{i\theta_m}. \tag{3}$$



Model: Kuramoto
Oscillators: 49
k: 0.00
K: 4.20
Mean Frequency: 0.5
Cluster q: 2
Forcing F: 8.00
☑ Display info. ☑ Show Colors
Speed: medium

FIG. 1. Screen shot of a simulation of the **Kuramoto** model.

The degree of synchrony is given by $|R| \in [0, 1]$, so that in an incoherent state $|R| \approx 0$ and in a very synchronized state $|R| \approx 1$.

A screenshot of a **Kuramoto** model simulation in **Synched** is shown in Fig. 1. R is plotted in the inner circle and the degree of synchrony $|R|$ is shown in the bottom left corner. Furthermore, when the color option is on, phase-locked oscillators are plotted in yellow and drifting oscillators are plotted in red.

II. TIME-DELAYS

The **Time-Delayed** model in **Synched** is based on the formulation in Reference [2], where each of N oscillators evolves according to

$$\dot{\theta}_n(t) = \omega_n + \frac{K}{N}\sum_{m=1}^{N}\sin[\theta_m(t - \tau_{n,m}) - \theta_n(t)]. \tag{4}$$

I.e., the effect of oscillator m on oscillator n is not felt instantaneously, but after some time delay τ_{nm}. As in the standard Kuramoto model, ω_n is the *intrinsic frequency* of oscillator n, which is drawn from the same distribution $g(\omega)$ and K is the global coupling strength. Furthermore, the delay τ_{nm} is drawn from a probability distribution $h(\tau)$. In **Synched**, we use

$$h(\tau) = \begin{cases} e^{-\tau} & \text{if } \tau \geq 0 \\ 0 & \text{otherwise.} \end{cases}. \tag{5}$$

Thus, K and Ω are again the two relevant parameters.

In addition to the standard order parameter R, this model

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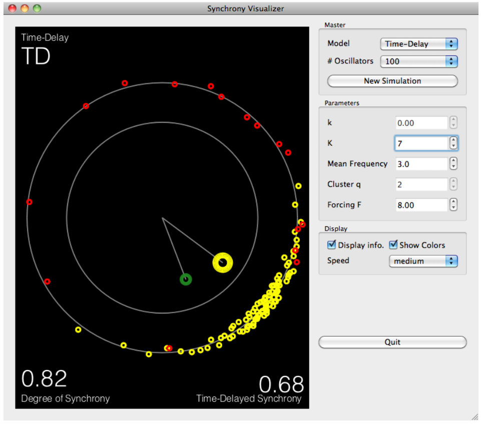
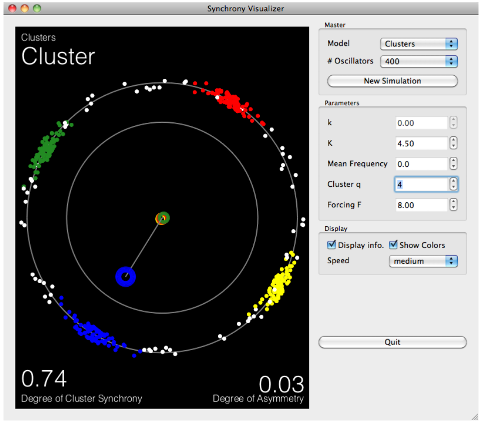


Model: Time-Delay
Oscillators: 100
k: 0.00
K: 7
Mean Frequency: 3.0
Cluster q: 2
Forcing F: 8.00
☑ Display info. ☑ Show Colors
Speed: medium
0.82 Degree of Synchrony
0.68 Time-Delayed Synchrony

FIG. 2. Screen shot of a simulation of the **Time-Delayed** model.



Model: Clusters
Oscillators: 400
k: 0.00
K: 4.50
Mean Frequency: 0.0
Cluster q: 4
Forcing F: 8.00
☑ Display info. ☑ Show Colors
Speed: medium
0.74 Degree of Cluster Synchrony
0.03 Degree of Asymmetry

FIG. 3. Screen shot of a simulation of the **Clusters** model.

has time-delayed order parameter

$$Z = \frac{1}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} e^{i\theta_m(t-\tau_{n,m})}, \tag{6}$$

which effectively couples all the oscillator together.

A screenshot of a **Time-Delayed** model simulation in **Synched** is shown in Fig. 2. Plotted in the inner circle are R as a larger circle and Z as a smaller circle in yellow and green, respectively, if the color option is on. The degree of synchrony and time-delayed synchrony, given by $|R|$ and $|Z|$ are shown in the bottom left and right corners. Again, locked oscillators are plotted in yellow and drifting oscillators are plotted in red.

III. CLUSTERS

The **Clusters** model is taken from Reference [3], and is given by the system

$$\dot{\theta}_n = \omega_n + \frac{K}{N} \sum_{m=1}^{N} \sin[q(\theta_m - \theta_n)]. \tag{7}$$

Again, ω_n is the *intrinsic frequency* of oscillator n, which is drawn from the same distribution $g(\omega)$ and K is the global coupling strength. In addition to K and Ω, this model has the parameter q, which is the clustering number and changes the function with which oscillators are coupled to one another. In **Synched**, q can be chosen to be any integer between one and ten.

To quantify synchrony, we use the generalized order parameters

$$R_l = \frac{1}{N} \sum_{m=1}^{N} e^{il\theta_m} \tag{8}$$

for $l = 1, \ldots, q$. The degree of cluster synchrony is given by R_q and the degree of asymmetry in the system is given by $\sum_{l=1}^{q-1} |R_l|/(q-1)$.

A screenshot of a **Clusters** model simulation in **Synched** is shown in Fig. 3. $R_1, \ldots, R_q$ are plotted in different colors in the inner circle (R_q is plotted with a larger circle than the other order parameters). The degree of cluster synchrony and degree of asymmetry are shown in the bottom left and right corners. When the color option is on locked oscillators are plotted in a color corresponding to the cluster it belongs to and drifting oscillators remain white.

IV. COMMUNITIES

The **Communities** is the most complicated model included in **Synched** and describes hierarchical synchrony in networks with community structure, which is studied in Reference [4]. It models a system with C communities, each containing N oscillators. Each oscillator evolves according to

$$\dot{\theta}_n^\sigma = \omega_n + \frac{k}{N} \sum_{m=1}^{N} \sin(\theta_m^\sigma - \theta_n^\sigma) + \frac{K}{CN} \sum_{\sigma' \neq \sigma} \sum_{m=1}^{N} \sin(\theta_m^{\sigma'} - \theta_n^\sigma), \tag{9}$$

where σ denotes the community, ω_n is the *intrinsic frequency* of oscillator n, which is drawn from the same distribution $g(\omega)$, and k and K denote the local and global coupling strengths. Like K, k can be chosen between zero and ten.

Each community has a local order parameter

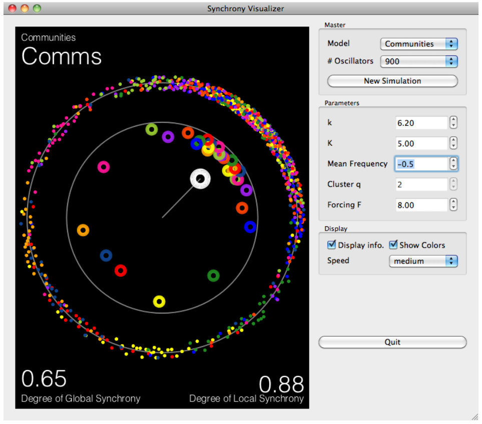
$$r_\sigma = \frac{1}{N} \sum_{m=1}^{N} e^{i\theta_m^\sigma}, \tag{10}$$

and the global order parameter is

$$R = \frac{1}{C} \sum_{\sigma=1}^{C} r_\sigma. \tag{11}$$

The degree of global synchrony is given by $|R|$ and the average degree of local synchrony is given by $\hat{r} = \sum_{\sigma=1}^{C} |r_\sigma|$.

A screenshot of a **Communities** model simulation in **Synched** is shown in Fig. 3. R is plotted as a large white circle in the inner circle, along with each r_σ in different colors. Along the outer circle individual oscillators are plotted in the color corresponding to their local order parameter r_σ, so that individual oscillators may be easily matched with their community.



Model: Communities
Oscillators: 900
k: 6.20
K: 5.00
Mean Frequency: -0.5
Cluster q: 2
Forcing F: 8.00
☑ Display info. ☑ Show Colors
Speed: medium
0.65 Degree of Global Synchrony
0.88 Degree of Local Synchrony

FIG. 4. Screen shot of a simulation of the **Communities** model.

V. FORCING

Finally, we discuss forcing. For each model described above, oscillators can be forced to a particular phase simply by clicking on the display. While the mouse is held, each oscillator will be forced to the phase of the location of the mouse with a strength proportional to the parameter F. This is done by adding the term $F\rho\sin(\phi - \theta_n)$ to each $\dot{\theta}_n$ ODE, where $\rho e^{i\psi}$ is the location of the mouse on the display.

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[1] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, New York, 1984).
[2] W. S. Lee, E. Ott, and T. M. Antonsen, Phys. Rev. Lett. **103**, 044101 (2009).
[3] P. S. Skardal, E. Ott, and J. G. Restrepo, Phys. Rev. E **84**, 036208 (2011).
[4] P. S. Skardal and J. G. Restrepo, Phys. Rev. E **85**, 016208 (2012).