Problem 1: Let $\lambda$ be a real number such that $\lambda \in (0, 1)$, and let $a$ and $b$ be two non-negative real numbers. Prove that
\begin{equation}
(1) \quad a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda) b,
\end{equation}
with equality iff $a = b$.

Hint: Consider the case $b = 0$ first. When $b \neq 0$, change variables to $t = a/b$.

Problem 2: [Hölder’s inequality] Suppose that $p$ is a real number such that $1 < p < \infty$, and let $q$ be such that $p^{-1} + q^{-1} = 1$. Let $(X, \mu)$ be a measure space, and suppose that $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$. Prove that $fg \in L^1(X, \mu)$, and that
\begin{equation}
(2) \quad ||fg||_1 \leq ||f||_p ||g||_q.
\end{equation}
Prove that equality holds iff $\alpha ||f||_p = \beta ||g||_q$ for some $\alpha, \beta$ such that $\alpha \beta \neq 1$.

Hint: Consider first the case where $||f||_p = 0$ or $||g||_q = 0$. For the case $||f||_p ||g||_q \neq 0$, use (1) with
\[
\begin{align*}
& a = \left| \frac{f(x)}{||f||_p} \right|^p, \\
& b = \left| \frac{g(x)}{||g||_q} \right|^q, \\
& \lambda = \frac{1}{p}.
\end{align*}
\]

Problem 3: [Minkowski’s inequality] Let $(X, \mu)$ be a measure space, and let $p$ be a real number such that $1 \leq p \leq \infty$. Prove that for $f, g \in L^p(X, \mu)$,
\begin{equation}
(3) \quad ||f + g||_p \leq ||f||_p + ||g||_p.
\end{equation}

Hint: Consider the cases $p = 1, \infty$ separately. For $p \in (1, \infty)$, note that
\[
|f(x) + g(x)|^p \leq (|f(x)| + |g(x)|) |f(x)| + g(x)|^{p-1}, \quad \forall x \in X.
\]
Then integrate both sides of (3) and apply (2) to the right hand side.