Homework set 5 — APPM5440

2.4: Let’s consider $X = [-1, 1]$ instead. Then set $f(x) = |x|$, and

$$f_n(x) = \frac{1 + nx^2}{\sqrt{n + n^2x^2}}.$$ 
Then $f_n \to f$ uniformly, $f_n \in C^\infty(X)$, and $f$ is not differentiable. (To justify the shift we made initially, simply note that if we define $g_n \in C([0, 1])$ by $g_n(y) = f_n(2y - 1)$, then $g_n$ is an answer to the original problem.)

2.5: Set $I = [a, b]$. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $C^1(I)$. Since

$$||f_n - f_m||_u \leq ||f_n - f_m||_{C^1},$$
the sequence $(f_n)$ is Cauchy in $C(I)$. Since $C(I)$ is complete, there exists a function $f \in C(I)$ such that $f_n \to f$ uniformly.

Next set $g_n = f'_n$. Then

$$||g_n - g_m||_u = ||f'_n - f'_m||_u \leq ||f_n - f_m||_{C^1},$$ 
so $(g_n)$ is Cauchy in $C(I)$. Therefore, there exists a function $g \in C(I)$ such that $g_n \to g$ uniformly.

It remains to prove that $f \in C^1(I)$, and that $f_n \to f$ in $C^1(I)$. Fix any $x \in I$, and any $h \in \mathbb{R}$ such that $x + h \in I$. Then

$$\frac{1}{h} (f(x + h) - f(x)) = \lim_{n \to \infty} \frac{1}{h} (f_n(x + h) - f_n(x))$$

$$= \lim_{n \to \infty} \frac{1}{h} \int_0^h f'_n(x + t) \, dt$$

$$= \lim_{n \to \infty} \frac{1}{h} \int_0^h g_n(x + t) \, dt.$$ 
Now recall that uniform convergence on a finite interval implies convergence of integrals. Since $g_n \to g$ uniformly, we therefore find that

$$\frac{1}{h} (f(x + h) - f(x)) = \frac{1}{h} \int_0^h g(x + t) \, dt.$$ 
Since $g$ is continuous, the limit as $h \to 0$ exists, and so

$$f'(x) = \lim_{h \to 0} \frac{1}{h} (f(x + h) - f(x)) = \lim_{h \to 0} \frac{1}{h} \int_0^h g(x + t) \, dt = g(x).$$
This proves that $f \in C^1(I)$. To prove that $f_n \to f$ in $C^1(I)$, we note that

$$||f - f_n||_{C^1} = ||f - f_n||_u + ||f' - f'_n||_u = ||f - f_n||_u + ||g - g_n||_u.$$ 
By the construction of $f$ and $g$, it follows that $||f - f_n||_{C^1(I)} \to 0$.

2.7: Set $I = [0, 1]$, and $\Omega = \{f \in C(I) : \text{Lip}(f) \leq 1, \int f = 0\}.$

We will use the Arzelà-Ascoli theorem, of course.
The Lipschitz condition implies that $\Omega$ is equicontinuous. (To prove this, fix any $\varepsilon > 0$. Set $\delta = \varepsilon$. Then for any $f \in \Omega$, and $|x - y| < \delta$, we have $|f(x) - f(y)| \leq \text{Lip}(f) |x - y| \leq |x - y| < \varepsilon$.)

To prove that $\Omega$ is bounded, note that if $\int f = 0$, and $f$ is continuous, then there must exist an $x_0 \in I$ such that $f(x_0) = 0$. Then for any $x \in I$ and any $f \in \Omega$, we have $|f(x)| = |f(x) - f(x_0)| \leq \text{Lip}(f) |x - x_0| \leq |x - x_0| \leq 1$. So $||f||_u \leq 1$.

Finally we need to prove that $\Omega$ is closed. Let $(f_n)$ be a Cauchy sequence in $\Omega$. Since $C(I)$ is complete, there exists an $f \in C(I)$ such that $f_n \to f$ uniformly. We need to prove that $f \in \Omega$. Since $f_n \to f$ uniformly, we know both that $\text{Lip}(f) \leq \limsup_{n \to \infty} \text{Lip}(f_n) \leq 1$, and that $\int f = \lim_{n \to \infty} \int f_n = 0$. This proves that $f \in \Omega$.

**2.8:** We will explicitly construct a dense countable subset $\Omega$ of $C([a, b])$. Without loss of generality, we can assume that $a = 0$ and that $b = 1$.

For $n = 1, 2, \ldots$, and for $j = 0, 1, 2, \ldots, n$, set $x_j^{(n)} = j/n$. Let $\Omega_n$ denote the subset of $C(I)$ of functions that (1) are linear on each interval $[x_{j-1}^{(n)}, x_j^{(n)}]$, and (2) take on rational values for each $x_j^{(n)}$. Since each function in $\Omega_n$ is uniquely defined by its values on the $x_j^{(n)}$’s, we can identify $\Omega_n$ by $\mathbb{Q}^n$. Hence $\Omega_n$ is countable.

Set $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Since each $\Omega_n$ is countable, $\Omega$ is countable.

It remains to prove that $\Omega$ is dense in $C(I)$. Fix any $f \in C(I)$, and any $\varepsilon > 0$. Since $I$ is compact, $f$ is uniformly continuous on $I$ so there exists a $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/5$. Pick an $n$ such that $1/n < \delta$, and pick a $\varphi \in \Omega_n$ such that $|\varphi(x_j^{(n)}) - f(x_j^{(n)})| < \varepsilon/5$ for $j = 0, 1, 2, \ldots, n$. We will prove that $||\varphi - f||_u < \varepsilon$: Fix an $x \in I$. Then pick $j \in \{1, 2, \ldots, n\}$ so that $x \in [x_{j-1}^{(n)}, x_j^{(n)}]$. Then

$$|f(x) - \varphi(x)| \leq |f(x) - f(x_j^{(n)})| + |f(x_j^{(n)}) - \varphi(x_j^{(n)})| + |\varphi(x_j^{(n)}) - \varphi(x)|.$$ 

The first term is bounded by $\varepsilon/5$ due to the uniform continuity of $f$. The second term is bounded by $\varepsilon/5$ by the selection of $\varphi$. For the third term, we find that

$$|\varphi(x_j^{(n)}) - \varphi(x)| \leq |\varphi(x_j^{(n)}) - \varphi(x_{j-1}^{(n)})|$$

$$\leq |\varphi(x_j^{(n)}) - f(x_j^{(n)})| + |f(x_j^{(n)}) - f(x_{j-1}^{(n)})| + |f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)})|.$$ 

The first and the last terms are bounded by $\varepsilon/5$ by the selection of $\varphi$, and the middle term is bounded by $\varepsilon/5$ by the uniform continuity of $f$. It follows that $|f(x) - \varphi(x)| < \varepsilon$. 


2.9: (a) Suppose that \( w(x) > 0 \) for \( x \in (0, 1) \). We will verify that \(|\cdot||_w\) is a norm:

(i) \(|\lambda f||_w = \sup_x w(x)|\lambda f(x)| = |\lambda| \sup_x w(x)|f(x)| = |\lambda||f||_w\).

(ii) \(|f + g||_w = \sup_x w(x)|f(x) + g(x)| \leq \sup_x w(x)(|f(x)| + |g(x)|) \leq \sup_x w(x)|f(x)| + \sup_x w(x)|g(x)| = |f||_w + |g||_w\).

(iii) If \( f = 0 \), then clearly \(|f||_w = 0\). Conversely, if \( f \neq 0 \), then \( f(x_0) \neq 0 \) for some \( x_0 \in (0, 1) \). Then \(|f||_w \geq w(x_0)|f(x_0)| > 0\).

(b) Assume that \( w(x) > 0 \) for \( x \in [0, 1] =: I \). Set \( m = \inf_{x \in I} w(x) \) and \( M = \sup_{x \in I} w(x) \). Since \( I \) is compact and \( w \) is continuous, \( w \) attains both its inf and its sup, and therefore \( m > 0 \) and \( M < \infty \). Then

\[
|f|_u = \sup_{x \in I} |f(x)| \geq \sup_{x \in I} \frac{w(x)}{M} |f(x)| = \frac{1}{M} ||f||_w.
\]

and

\[
|f|_u = \sup_{x \in I} |f(x)| \leq \sup_{x \in I} \frac{w(x)}{m} |f(x)| = \frac{1}{m} ||f||_w.
\]

It follows that

\[
\frac{1}{M} ||f||_w \leq ||f||_u \leq \frac{1}{m} ||f||_w.
\]

(c) Set \(|||f||| = \sup_{x \in I} |x f(x)|\). We will prove that \(||\cdot||\) is not equivalent to the uniform norm. Set for \( n = 1, 2, \ldots \)

\[
f_n(x) = \begin{cases} 1 & x \in [0, 1/n], \\ 0 & x \in (1/n, 1]. \end{cases}
\]

Then

\[
\inf_{||f||=1} ||f|| \leq \inf_n |||f_n||| = \inf \frac{1}{n} = 0.
\]

This proves that there cannot exist a \( c > 0 \) such that \(|||f||| \geq c||f||\).

(d) We will prove that the set \( C(I) \) equipped with the norm \(||\cdot||\) is not a Banach space by constructing a Cauchy sequence with no limit point in \( C(I) \). For \( n = 1, 2, \ldots \), define \( f_n \in C(I) \) by

\[
f_n(x) = \begin{cases} x^{-1/2} & x \in (1/n, 1], \\ \sqrt{n} & x \in [0, 1/n]. \end{cases}
\]

Fix a positive integer \( N \). Then, if \( m, n \geq N \), we have

\[
|||f_n - f_m||| = \sup_{x \in [0, 1/N]} |f_n(x) - f_m(x)|
\]

\[
\leq \sup_{x \in [0, 1/N]} (|f_n(x)| + |f_m(x)|)
\]

\[
\leq \sup_{x \in [0, 1/N]} (x \sqrt{x} + x \sqrt{x}) = 2N^{-1/2}.
\]

Consequently, \( (f_n)_{n=1}^\infty \) is a Cauchy sequence. But \( f_n \) cannot converge uniformly to any function in \( C(I) \). (To prove the last contention, suppose that
\( f_n \to f \) for some \( f \in C(I) \). Then \( f(0) = \lim_{n \to \infty} f_n(0) = \infty \), which is a contradiction.)

**Problem 1:** Let \( X = [0, \infty) \). Construct a sequence of functions \( f_n : X \to \mathbb{R} \) that converges uniformly (and hence pointwise), but that does not converge in \( L^2(X) \).

**Solution:** One possible choice is
\[
\varphi_n(x) = \begin{cases} 
  n^{-1/2} & x \in [0, n], \\
  0 & x \in (n, \infty).
\end{cases}
\]
Then \( \varphi_n \to 0 \) uniformly, but \( \|\varphi_n - 0\| = 1 \) for all \( n \).

**Problem 2:** Let \( X = [0, 1] \). Construct a sequence of functions \( f_n : X \to \mathbb{R} \) that converges in \( L^2(X) \) but such that the sequence of numbers \( (f_n(x))_{n=1}^{\infty} \) does not converge for any \( x \in X \).

**Solution:** For \( I = [a, b] \) an interval in \( X \), consider the function
\[
\chi_I(x) = \begin{cases} 
  1 & x \in I, \\
  0 & x \in X \setminus I.
\end{cases}
\]
Now construct intervals \( I_n \) that (1) decrease in size, and (2) march across the interval \([0, 1]\). For instance,
\[
\begin{align*}
I_1 &= [0/2, 1/2], \\
I_2 &= [1/2, 2/2], \\
I_3 &= [0/4, 1/4], \\
I_4 &= [1/4, 2/4], \\
I_5 &= [2/4, 3/4], \\
I_6 &= [3/4, 4/4], \\
I_7 &= [0/8, 1/8], \\
I_8 &= [1/8, 2/8], \\
I_9 &= [2/8, 3/8], \\
&\vdots
\end{align*}
\]
Set \( \varphi_n = \chi_{I_n} \). Then \( \varphi_n \to 0 \) in \( L^2 \), but for any fixed \( x \), the sequence of numbers \( (\varphi_n(x))_{n=1}^{\infty} \) does not converge.