Ex 1.2. \[
\sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}
\]

So

\[
\left| \sum_{n=0}^{N} x^n - \frac{1}{1-x} \right| \leq \frac{1 \cdot x^{N+1}}{1 - |x|}
\]

\[
\rightarrow 0 \quad \text{as } N \rightarrow \infty
\]

for \( |x| < 1 \). Note that \( \frac{1 \cdot x^{N+1}}{1 - |x|} \) is monotone decreasing.

Therefore, for \( \varepsilon > 0 \), pick \( \delta > 0 \) such that \( \frac{\delta^{N+1}}{1 - \delta} < \varepsilon \).
1.3 Start from the triangle inequality:

\[ d(x,y) + d(y,z) \geq d(x,z) \]

Therefore,

\[ d(x,y) \geq d(y,z) - d(x,z) \]

By interchanging the roles of \( x \) and \( y \), we also get

\[ d(x,y) \geq d(x,z) - d(y,z) \]

So

\[ d(x,y) \geq |d(x,z) - d(y,z)| \]
Ex 1.5 Suppose \((X, \| \cdot \|)\) is a normed linear space. Prove that \(d(x,y) = \| x - y \|\) defines a metric on \(X\).

**Proof:** Due to the properties of a norm, we immediately have \(d(y,x) = d(x,y) \geq 0\) and \(d(x,y) = 0\) iff \(x = y\). The triangle inequality follows from the triangle inequality for the norm:

\[
d(x,y) = \| x - y \| = \| (x - z) + (z - y) \|
\]

\[
\leq \| x - z \| + \| z - y \|
\]

\[
\leq d(x,z) + d(z,y)
\]

- Prove that \(d_f(x,y) = \frac{\| x - y \|}{1 + \| x - y \|}\) defines a metric on \(X\).

**Proof:** We will prove a more general result. Suppose \(f : [0, \infty) \rightarrow [0, \infty)\) has the following properties:

(i) \(f(0) = 0\), \(f(x) > 0\) for \(x > 0\)

(ii) \(f\) is non-decreasing

(iii) \(f\) is concave (so \(f'\) is non-increasing)

**Claim** Then, for any metric \(d\) on \(X\),

\[
d_f(x,y) = f(d(x,y))\]

also defines
a metric on \( X \).

It is easy to check, using calculus, that \( f(x) = \frac{x}{1+x} \) satisfies (i) - (iii).

**Proof of the claim:**

Again, the only non-trivial property is the triangle inequality. First note that that \( f \) has the property that for any \( a, b \geq 0 \)

\[
f(a+b) \leq f(a) + f(b)
\]

To see this, use property (iii) as follows:

\[
f(a+b) = f(a) + f(a+b) - f(a)_{a+b}
\]

\[
= f(a) + \int_{a}^{a+b} f'(t) dt
\]

\[
= f(a) + \int_{0}^{b} f'(a+s) ds
\]

\[\text{(iii)}\]

\[
\leq f(a) + \int_{0}^{b} f'(s) ds
\]

\[
\leq f(a) + f(b) - f(a)
\]

\[\text{(i)}\]

\[
= f(a) + f(b)
\]

Now, we can prove the triangle inequality for \( f_x \):

\[
d_f(x, y) = f(d(x, y)) \leq f(d(x, z) + d(z, y))
\]

\[
\leq f(d(x, z)) + f(d(z, y)) = d_f(x, z) + d_f(z, y)
\]
1.9 Let \((x_n)\) be a bounded sequence in \(\mathbb{R}\).

(a) Prove that for every \(\epsilon > 0\) and every \(N \in \mathbb{N}\)
\[ n_1, n_2 \geq N \quad \text{s.t.} \]
\[ \limsup_n x_n \leq x_{n_1} + \epsilon \]
\[ \liminf_n x_n \geq x_{n_2} - \epsilon \]

**Proof:** As the \(\limsup\) and \(\liminf\) depend only on the tail of the sequence, WLOG we can assume that \(N=1\) (i.e. if we prove it for \(N=1\), we can apply the result to the sequence \(y_n = x_{n+N}\)).

The second inequality follows from the first by applying it to the sequence \((-x_n)\) and seeing that
\[ \limsup_n -x_n = -\liminf_n x_n \]

So, let us prove the first inequality. Suppose such \(n_1, n_2\) did not exist, then
\[ x_{n_1} < \limsup_n x_n - \epsilon \]
for all \(n_1\), and hence \(\limsup_n x_n - \epsilon\) would be an upper bound for the
sequence, implying the contradiction
\[ \limsup_{n} x_n \leq \limsup_{n} x_n - \varepsilon \]
As \( x_n \) is assumed to be bounded, this is impossible.

(b) Prove that for every \( \varepsilon > 0 \) there is \( N \in \mathbb{N} \)
\[ x_m \leq \limsup_{n} x_n + \varepsilon \quad \forall \, m \geq N \]
and
\[ x_m \geq \liminf_{n} x_n - \varepsilon \quad \forall \, m \geq N \]

Proof. Again, the second inequality follows from the first by considering the sequence \((-x_n)\). The first inequality follows immediately from the definition of \( \limsup \):
\[ \limsup_{n} x_n = \lim_{m \to \infty} \sup_{k \geq m} x_k \]
it follows that \( \forall \, \varepsilon > 0 \) \( \exists \, N \) such that
\[ |(\limsup_{n} x_n) - \sup_{k \geq m} x_k| < \varepsilon \quad \forall \, m \geq N \]. A fortiori
\[ x_m \leq \sup_{k \geq m} x_k \leq \varepsilon + \limsup_{n} x_n \]
(c) First, suppose \( x_n \) converges, say to \( x \). Then, from part (a) we can find a subsequence \( x_{n_k} \) such that
\[
\limsup_{n \to \infty} x_n \leq x_{n_k} + \varepsilon + k
\]
As \( x_n \to x \), also \( x_{n_k} \to x \). Therefore,
\[
\limsup_{n \to \infty} x_n \leq x + \varepsilon
\]
Similarly,
\[
x - \varepsilon \leq \liminf_{n \to \infty} x_n
\]
So
\[
x - \varepsilon \leq \liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n \leq x + \varepsilon
\]
As \( \varepsilon > 0 \) is arbitrary, this implies
\[
\limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = x.
\]
Second, suppose \( \limsup_{n \to \infty} x_n = \liminf_{n \to \infty} x_n = 1 \).
Then, from part (b), we see that
\[ x_m \to L \]

By the definition of convergence of a sequence this means:

\[ |x_m - L| < \varepsilon. \]
10. First, we show that if
\[ a_n \leq b_n \quad \text{for all } n \in \mathbb{N} \]
then
\[ \limsup_{n \to \infty} a_n \leq \limsup_{n \to \infty} b_n . \]

**Proof:** Given \( \varepsilon > 0 \), there is an \( N \in \mathbb{N} \)
such that
\[ b_n \leq \limsup_{n \to \infty} b_n + \varepsilon \quad \forall \ n \geq N, \]
and there is an \( n_1 \geq N \) such that
\[ \limsup_{n \to \infty} a_n \leq a_{n_1} + \varepsilon . \]

Hence
\[ \limsup_{n \to \infty} a_n \leq a_{n_1} + \varepsilon \]
\[ \leq b_{n_1} + \varepsilon \]
\[ \leq \limsup_{n \to \infty} b_n + 2 \varepsilon . \]

Since \( \varepsilon > 0 \) is arbitrary, the result follows. \( \square \)
For every $\alpha \in A$, we have

$$\inf_{\beta \in A} x_{n, \beta} \leq x_{n, \alpha}.$$  

Therefore, by the previous result,

$$\limsup_{n \to \infty} \left( \inf_{\beta \in A} x_{n, \beta} \right) \leq \limsup_{n \to \infty} x_{n, \alpha}.$$  

It follows that $\limsup_{n \to \infty} \left( \inf_{\beta \in A} x_{n, \beta} \right)$ is a lower bound of the set $\{ \limsup_{n \to \infty} x_{n, \alpha} \mid \alpha \in A \}$, so

$$\limsup_{n \to \infty} \left( \inf_{\beta \in A} x_{n, \beta} \right) \leq \inf_{\alpha \in A} \left( \limsup_{n \to \infty} x_{n, \alpha} \right).$$

The corresponding result for the liminf of sup follows by application of this result to $\{-x_{n, \alpha} \}$.
Example of strict inequality

Let $A = \mathbb{N}$, and define

$$x_{n,m} = \begin{cases} 0 & n \leq m \\ 1 & n > m \end{cases}$$

Then

$$\limsup_{n \to \infty} x_{n,m} = 1 \quad \text{for all } m \in \mathbb{N}$$

$$\inf_{m \in \mathbb{N}} x_{n,m} = 0 \quad \text{for all } n \in \mathbb{N}$$

So

$$\limsup_{n \to \infty} (\inf_{m \in \mathbb{N}} x_{n,m}) = 0$$

$$\inf_{m \in \mathbb{N}} (\limsup_{n \to \infty} x_{n,m}) = 1$$

1.12 Let \( f : X \to Y \), \( g : Y \to Z \) be continuous functions. Show that \( h : X \to Z \),
\( h = g \circ f \) is also continuous.

Proof. The shortest proof is by applying Proposition 1.46. So, we need to argue
that for any \( G \subset Z \), open,
\( h^{-1}(G) \) is open. This follows
by two more applications of
Proposition 1.46:

\[
h^{-1}(G) = f^{-1}(g^{-1}(G))
\]

which is open by the continuity of
\( g \) and \( f \).
First, we show that \((X = \text{metric space})\)
\[d(\cdot, E) : X \to \mathbb{R}\]
is continuous. If \(x_n \to x\), then
\[
d(x, E) = \inf_{y \in E} d(x, y)
\]
\[
= \inf_{y \in E} \lim_{n \to \infty} d(x_n, y)
\]
\[
\geq \limsup_{n \to \infty} \left[ \inf_{y \in E} d(x_n, y) \right]
\]
\[
\geq \limsup_{n \to \infty} d(x_n, E).
\]
So \(d(\cdot, E)\) is upper semicontinuous.

To prove that \(d(\cdot, E)\) is also lower semicontinuous (and hence continuous), we let \(\varepsilon > 0\). If \(x_n \to x\), there exists \(y_n \in E\) such that
\[
d(x_n, y_n) < d(x_n, E) + \varepsilon
\]
\[
\Rightarrow d(x, y_n) \leq d(x, x_n) + d(x_n, y_n)
\]
\[
\Rightarrow d(x, y_n) \leq d(x, x_n) + d(x_n, E) + \varepsilon
\]
Letting $n \to \infty$, we get
\[ d(x, E) \leq \liminf_{n \to \infty} d(x_n, E) + \varepsilon. \]
Since this inequality holds for all $\varepsilon > 0$, we conclude that $d(\cdot, E)$ is lower semicontinuous.

If $F$ is closed, then $d(x, F) = 0$ if and only if $x \in F$: if $d(x, F) = 0$ then there exist $x_n \in F$ such that $d(x, x_n) \to d(x, F) = 0$; hence $x_n \to x$ and $x \in F$ since $F$ is closed.

Since $F \cap G^c = \emptyset$ and $F, G^c$ are closed, we have
\[ d(x, F) + d(x, G^c) \neq 0 \quad \forall x \in X. \]

Hence
\[ f(x) = \frac{d(x, G^c)}{d(x, F) + d(x, G^c)} \]
is continuous, and
\[ f(x) = \begin{cases} 0 & x \notin F \\ 1 & x \in F \end{cases} \]