Applied Analysis (APPM 5440): Midterm 1

Problem 1: No motivation required for (a) and (c). Only brief motivations required for (b) and (d). 2 points each:
(a) Define what it means for a metric space \((X, d)\) to be complete.
(b) Set \(X = [0, 1] \cup [1, 2]\), and \(\Omega = [0, 1]\). Is \(\Omega\) open in the metric space \((X, | \cdot |)\)?
(c) For \(n \in \mathbb{N}\), set \(x_n = e^{-1/n}(1 + (-1)^n) - 1/n\). Give numerical values for the quantities that exist among: \(\lim_{n \to \infty} x_n\), \(\limsup_{n \to \infty} x_n\), and \(\liminf_{n \to \infty} x_n\).
(d) Construct a sequence \((x_n)_{n=1}^{\infty}\) such that \(0 \leq x_n \leq 1\) for every \(n\), and such that for any \(\alpha \in [0, 1]\), there exists a subsequence \((x_{n_j})_{j=1}^{\infty}\) such that \(x_{n_j} \to \alpha\) as \(j \to \infty\).

Problem 2: Define a norm on \(\mathbb{R}^d\) by setting, for \(x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d\),
\[||x|| = \sum_{1 \leq j \leq d} |x_j|\]
Using the fact that \((\mathbb{R}, | \cdot |)\) is complete, prove that \((\mathbb{R}^d, || \cdot ||)\) is complete. (3p)

Problem 3: Let \((X, d_X), (Y, d_Y),\) and \((Z, d_Z)\) denote metric spaces, and let \(f : X \to Y\), and \(g : Y \to Z\) denote continuous functions. Prove that the function \(h : X \to Z\) that is defined by \(h(x) = g(f(x))\) is continuous. (3p)

Problem 4: Let \(X\) denote the set of real numbers, and equip \(X\) with the discrete metric \(d_X\) (so that \(d_X(x, y) = 0\) if \(x = y\), and \(d_X(x, y) = 1\) otherwise). Let \((Y, d_Y)\) denote another metric space. For each statement below, either prove that it is necessarily true, or give a counter-example. (2p each.)
(a) Let \(f\) be a function from \((X, d_X)\) to \((Y, d_Y)\). Then \(f\) is necessarily continuous.
(b) Let \(g\) be a function from \((Y, d_Y)\) to \((X, d_X)\). Then \(g\) is necessarily continuous.

Problem 5: Let \((X, d)\) denote a metric space, and let \(Y\) denote a subset of \(X\). Consider the following three sets, and three statements:
\(\Omega_1\) is the set of all \(x \in X\) for which there exists \((y_n)_{n=1}^{\infty} \subseteq Y\) such that \(y_n \to x\).
\(\Omega_2 = \bigcap_{\alpha \in A} F_\alpha\) where \(\{F_\alpha\}_{\alpha \in A}\) is the set of all closed sets in \((X, d)\) that contain \(Y\).
\((\hat{Y}, \hat{d})\) is the completion of the metric space \((Y, d)\).
(a) \(\Omega_1 \subseteq \Omega_2\)
(b) \(\Omega_2 \subseteq \Omega_1\)
(c) The two metric spaces \((\Omega_2, d)\) and \((\hat{Y}, \hat{d})\) are isometrically isomorphic.
For each statement, either prove that it is necessarily true, or give a counter-example (if you give a counter-example, you do not need to justify it in detail). You may not use any theorems given in class that relate to the concept of “closure”. (2p each.)