Problem 1: No motivation required. 2p each:

(a) Let \((X, T)\) denote a topological space. Specify the axioms that \(T\) must satisfy.

(b) Let \((X, T)\) denote a topological space. Define what it means for \(T\) to be Hausdorff.

(c) Let \((X, T)\) denote a topological space, let \((x_n)_{n=1}^{\infty}\) denote a sequence in \(X\), and let \(x\) denote an element of \(X\). Define what it means for \(x_n\) to converge to \(x\). (\(T\) is not necessarily metrizable.)

Problem 2: Consider the set \(X = \{a, b, c\}\), and the collection of subsets \(T = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}\). Is \(T\) a metrizable topology? List the compact subsets of \(X\). Give an example of a function \(f: X \rightarrow \mathbb{R}\) that is continuous, and one example of a function \(g: X \rightarrow \mathbb{R}\) that is not. Justify your answers briefly. (6p)

Problem 3: Let \(X\) denote the set of all continuous functions on the interval \(I = [-\pi, \pi]\). Equip \(X\) with the norm

\[ ||f|| = \int_{-\pi}^{\pi} |f(y)| \, dy. \]

Consider the operator \(T \in \mathcal{B}(X)\) that is defined by

\[ [Tf](x) = \int_{0}^{\pi} \sin(x) y^2 f(y) \, dy. \]

Calculate the norm of \(T\) in \(\mathcal{B}(X)\). (4p total: 2p for the correct answer \(\alpha\), and 1p each for the proofs that \(\alpha \leq ||T||\) and that \(\alpha \geq ||T||\).)

Problem 4: Let \(X\) be a Banach space with a compact subset \(K\). Suppose that \((x_n)_{n=1}^{\infty}\) is a sequence of elements in \(K\) that converges weakly to some element \(x \in K\). Is it necessarily the case that the sequence also converges in norm to \(x\)? Either prove that this is the case, or give a counter-example. (4p)

Problem 5: Consider the Banach space \(X = l^2(\mathbb{N})\), and the operator \(T \in \mathcal{B}(X)\) defined by

\[ Tx = (\frac{1}{1} x_1, \frac{1}{2} x_2, \frac{1}{3} x_3, \ldots). \]

Prove that \(\text{ran}(T)\) is not topologically closed. (4p)