Problem 1: No motivation required. 2p each:

(a) Let $(X, T)$ denote a topological space. Specify the axioms that $T$ must satisfy.

(b) Let $(X, T)$ denote a topological space. Define what it means for $T$ to be Hausdorff.

(c) Let $(X, T)$ denote a topological space, let $(x_n)_{n=1}^{\infty}$ denote a sequence in $X$, and let $x$ denote an element of $X$. Define what it means for $x_n$ to converge to $x$. ($T$ is not necessarily metrizable.)

Solution: Check textbook.
Problem 2: Consider the set $X = \{a, b, c\}$, and the collection of subsets $T = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Is $T$ a metrizable topology? List the compact subsets of $X$. Give an example of a function $f : X \to \mathbb{R}$ that is continuous, and one example of a function $g : X \to \mathbb{R}$ that is not. Justify your answers briefly. (6p)

Solution: $T$ is a topology, but it is not metrizable. To prove this, we assume that there exists a metric $d$ that generates $T$. Set $\varepsilon = \min(d(b, a), d(b, c))$. Then $\{b\} = B_{\varepsilon/2}(b)$ so $\{b\}$ should be an open set. However, $\{b\} \notin T$.

Every subset of $X$ is compact (since $T$ is finite, every open cover of any subset is itself finite). Thus the compact sets are 
\[ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}. \]

The function $f$ defined by $f(x) = 1$ for $x = a, b, c$ is continuous. To prove this, let $G$ be an open subset of $\mathbb{R}$. If $1 \in G$, then $f^{-1}(G) = X$ which is an open set. If $1 \notin G$, then $f^{-1}(G) = \emptyset$ which is also open.

The function $g$ defined by 
\[ g(a) = 0, \quad g(b) = 0, \quad g(c) = 1 \]

is not continuous. To prove this, consider the open set $G = (1/2, 3/2)$ in $\mathbb{R}$. Then $g^{-1}(G) = \{c\}$ which is not an open set in $T$. 
Problem 3: Let $X$ denote the set of all continuous functions on the interval $I = [-\pi, \pi]$. Equip $X$ with the norm
\[ ||f|| = \int_{-\pi}^{\pi} |f(y)| \, dy. \]
Consider the operator $T \in \mathcal{B}(X)$ that is defined by
\[ [Tf](x) = \int_{0}^{\pi} \sin(x) y^2 f(y) \, dy. \]
Calculate the norm of $T$ in $\mathcal{B}(X)$. (4p total: 2p for the correct answer $\alpha$, and 1p each for the proofs that $\alpha \leq ||T||$ and that $\alpha \geq ||T||$.)

Solution: We have
\[ ||Tf|| = \int_{-\pi}^{\pi} \left| \int_{0}^{\pi} \sin(x) y^2 f(y) \, dy \right| \, dx = \int_{-\pi}^{\pi} |\sin(x)| \, dx \left| \int_{0}^{\pi} y^2 f(y) \, dy \right| \]
\[ = 4 \left| \int_{0}^{\pi} y^2 f(y) \, dy \right| \leq 4 \left( \sup_{y \in I} y^2 \right) \int_{0}^{\pi} |f(y)| \, dy \leq 4 \pi^2 ||f||. \]
It follows that $||T|| \leq 4 \pi^2$.

To prove that $||T|| \geq 4 \pi^2$, pick\(^1\) non-negative functions $f_n \in X$ such that $||f_n|| = 1$ and $\text{supp}(f) \subseteq [-\pi - 1/n, \pi]$. Then
\[ ||T|| = \sup_{||f||=1} ||Tf|| \geq \sup_{n} ||Tf_n|| = \sup_{n} \int_{-\pi}^{\pi} |\sin(x)| \, dx \int_{0}^{\pi} y^2 f_n(y) \, dy \]
\[ = \sup_{n} 4 \int_{-1/n}^{\pi} y^2 f_n(y) \, dy \geq \sup_{n} 4 \left( \inf_{y \in [-\pi - 1/n, \pi]} y^2 \right) \int_{-1/n}^{\pi} f_n(y) \, dy = \sup_{n} 4 \left( \pi - 1/n \right)^2 = 4 \pi^2. \]

\(^1\)In your solutions, drawing a picture of such a sequence is fine. An explicit formula is not required, but if you insist on one, consider
\[ f_n(x) = \begin{cases} 0 & x \in [-\pi, \pi - 1/n] , \\ 2n^2 (x - (\pi - 1/n)) & x \in (\pi - 1/n, \pi]. \end{cases} \]
Problem 4: Let $X$ be a Banach space with a compact subset $K$. Suppose that $(x_n)_{n=1}^\infty$ is a sequence of elements in $K$ that converges weakly to some element $x \in K$. Is it necessarily the case that the sequence also converges in norm to $x$? Either prove that this is the case, or give a counter-example. (4p)

Solution: The answer is yes. Suppose that the sequence $(x_n)_{n=1}^\infty$ satisfies the assumptions of the problem, but does not converge in norm to $x$. Then there exists an $\varepsilon > 0$, and a subsequence $(x_{n_j})_{j=1}^\infty$ such that

\[ ||x - x_{n_j}|| \geq \varepsilon, \quad \text{for } j = 1, 2, 3, \ldots \]

However, since $(x_{n_j})$ is a sequence in a compact set, it has a subsequence $(x_{n_{j_k}})_{k=1}^\infty$ that converges in norm. Since $x_{n_{j_k}} \rightharpoonup x$, this element must be $x$, which is impossible in view of (1).
Problem 5: Consider the Banach space $X = l^2(\mathbb{N})$, and the operator $T \in B(X)$ defined by

$$Tx = \left( \frac{1}{1}x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots \right).$$

Prove that $\text{ran}(T)$ is not topologically closed. (4p)

Solution: We know that a one-to-one operator has closed range if and only if it is coercive. We will prove that $T$ is one-to-one, but not coercive.

To see that $T$ is one-to-one, simply note that if $Tx = 0$, then clearly $x$ must be zero.

Next we prove that $T$ is not coercive. Let $e^{(n)}$ denote the canonical basis vectors,

$$e^{(1)} = (1, 0, 0, 0, \ldots),$$
$$e^{(2)} = (0, 1, 0, 0, \ldots),$$
$$e^{(3)} = (0, 0, 1, 0, \ldots),$$
$$\vdots$$

We have

$$||T e^{(n)}|| = \left| \frac{1}{n} e^{(n)} \right| = \frac{1}{n} ||e^{(n)}||$$

so there can exist no $c > 0$ such that $||Tx|| \geq c ||x||$ for all $x$.

Alternative solution: We will prove that the element

$$y = \left( \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right) \in X$$

belongs to $\text{ran}(T)$, but not to $\text{ran}(T)$. This proves that $\text{ran}(T)$ is not closed.

To prove that $y \in \overline{\text{ran}(T)}$, consider the elements $x^{(n)} \in X$ defined by

$$x^{(1)} = (1, 0, 0, 0, \ldots),$$
$$x^{(2)} = (1, 1, 0, 0, \ldots),$$
$$x^{(3)} = (1, 1, 1, 0, \ldots),$$
$$\vdots$$

Set $y^{(n)} = Tx^{(n)}$ so that $y^{(n)} \in \text{ran}(T)$. Since $y^{(n)} \rightarrow y$, it follows that $y \in \overline{\text{ran}(T)}$.

To prove that $y \not\in \text{ran}(T)$, note that if $Tx = y$, then $x = (1, 1, 1, \ldots)$ which is not an element of $X$. 