THE CONTRACTION MAPPING PRINCIPLE

Defn. Let \( X \) be a metric space, and \( f: X \rightarrow X \). We say that \( f \) is a contraction (mapping) if there exists an \( \alpha \) s.t. \( 0 < \alpha < 1 \) and
\[
d(f(x), f(y)) \leq \alpha d(x, y) \quad \forall x, y \in X.
\]

Alt. defn: \( \exists \alpha \in (0, 1) \) s.t. \( \forall x \in X \), \( \exists r \in \mathbb{R}^+ : f(B_r(x)) \subseteq B_{\alpha r}(f(x)) \).

Note: Obviously, every contraction is continuous.
(In fact, uniformly continuous.)

Defn. Let \( X \) be any set and \( f: X \rightarrow X \). We say that \( x_0 \) is a fixed point of \( f \) if \( f(x_0) = x_0 \).

Thm. Suppose that \( X \) is a complete metric space and that the map \( f: X \rightarrow X \) is a contraction. Then \( f \) has a unique fixed point in \( X \).

Proof. First we prove existence.
Let \( x_0 \) be an arbitrarily chosen point in \( X \). Set \( x_1 = f(x_0) \), \( x_2 = f(f(x_0)) \), \ldots, \( x_n = f(x_{n-1}) \).
We will prove that \( (x_n)_{n=1}^{\infty} \) is a Cauchy seq in \( X \).
Pick positive integers \( m \, \& \, n \) s.t. \( m > n \).
Proof contd.

Then \( d(x_n, x_n) = d(\mathbf{F}(x_{n-1}), \mathbf{F}(x_{n-1})) \leq \alpha d(x_{n-1}, x_{n-1}) d(\mathbf{F}(x_{n-2}), \mathbf{F}(x_{n-2})) \leq \alpha^2 d(x_{n-2}, x_{n-2}) \leq \cdots \leq \alpha^m d(x_{n-m}, x_0) \leq \alpha^m (d(x_{n-m}, x_{n-m}) + d(x_{n-m-1}, x_{n-m-2}) + \cdots + d(x_1, x_0)) \leq \alpha^m (\alpha^{n-m-1} d(x_1, x_0) + \alpha^{n-m-2} d(x_1, x_0) + \cdots + d(x_1, x_0)) \)

\[ = \alpha^m \frac{1 - \alpha^{n-m}}{1 - \alpha} d(x_1, x_0) = \frac{\alpha^m}{1 - \alpha} d(x_1, x_0) \to 0 \text{ as } m \to \infty. \]

Since \( I \) is complete & \( (x_n)_{n=1}^\infty \) is Cauchy \( \exists \) \( x \) s.t. \( x_n \to x \).

If \( f \) is cont \( \implies f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_n = x \) so \( x \) is a fixed point.

It remains to prove uniqueness. Suppose that \( x = f(x) \) & \( y = f(y) \).

Then \( d(x, y) = d(f(x), f(y)) \leq \alpha d(x, y) \implies d(x, y) = 0 \implies x = y. \)

**Thm.** Suppose that \( I \) is an interval in \( \mathbb{R} \) and that \( \tau_0 \in I \).

Suppose that \( \mathbf{F} : I \times \mathbb{R}^n \to \mathbb{R}^n \) is a continuous function s.t.

\[ |\mathbf{F}(t, u) - \mathbf{F}(t, v)| \leq C |u - v| \quad \forall \, u, v \in \mathbb{R}^n \quad \text{t.e} I \]

for some finite number \( C \). Then the eqn

\[
\begin{cases}
\mathbf{u}(t) = \mathbf{F}(t, \mathbf{u}(t)) & t \in I \\
\mathbf{u}(\tau_0) = \mathbf{u}_0
\end{cases}
\]

has for any \( \mathbf{u}_0 \in \mathbb{R}^n \) a unique soln on \( I \).

**Proof:** Fix a \( \tau_0 \in I \) and consider the eqn

\[ (*) \begin{cases}
\mathbf{u}(t) = \mathbf{F}(t, \mathbf{u}(t)) \\
\mathbf{u}(\tau_0) = \mathbf{u}_0
\end{cases} \]

Note that \( \mathbf{u} \) solves \( (*) \) iff

\[ \mathbf{u}(t) = \mathbf{u}_0 + \int_{\tau_0}^{t} \mathbf{F}(s, \mathbf{u}(s)) \, ds \quad (**) \]

Define \( F \) by setting...
Now fix $\varepsilon > 0$ and consider the map

$$F: C([t_0, t + \varepsilon]) \to C([t_0, t + \varepsilon]): u \mapsto \left[ F(u) \right](t) = u_t + \int_t^{t + \varepsilon} f(s, u(s)) ds.$$

We will prove that if $\varepsilon$ is small enough, then $F$ is a contraction:

$$\| F(u) - F(v) \| = \sup_{t, s} \left| \int_t^{t + \varepsilon} \left[ f(s, u(s)) ds - f(s, v(s)) ds \right] \right| \leq \sup_{t} \int_t^{t + \varepsilon} |f(s, u(s)) - f(s, v(s))| ds \leq C S \| u - v \|$$

We see that if $S < \frac{1}{10}$, then $F$ is a contraction on $C([t_0, t + \varepsilon])$. This implies that $(\ast\ast)$, and thus $(\ast)$, has a unique solution on $[t_0, t + \varepsilon]$.

Suppose that $I = [a, b]$, and $\Delta I = [a, s] \cup [s, t] \cup [t, b]$. By splitting the interval $[a, b]$ into pieces of length $S$, and repeating the existence proof, we prove existence on $[t_0, b]$. By "going backwards", we similarly prove existence on $[a, t_0]$.

Remark: The restriction to first-order ODE's is non-essential, since any higher-order ODE can be rewritten to a first-order ODE.

Example: (1) \begin{align*}
    \dot{u} &= f(t, u, u) \\
    u(t_0) &= a \\
    u(t_b) &= b
\end{align*}

Set $v_1 = u$ and $v_2 = \dot{u}$. Then (1) is equivalent to

(2) \begin{align*}
    \dot{v} &= \begin{bmatrix}
        f(t, v_1, v_2) \\
        f(t, v_1, v_2)
    \end{bmatrix} \\
    v(t_0) &= \begin{bmatrix}
        a \\
        b
    \end{bmatrix}
\end{align*}
Remark. The theorem applies to all linear ODE's
\[
\begin{align*}
\dot{u}(t) &= A(t)u(t) + b(t) \\
u(t_0) &= u_0
\end{align*}
\]
as long as \( C' = \sup_{t \in I} \sup_{\|\varphi\| = 1} |A(t)\varphi| \) is finite.

In this case \( |\varphi(t, u) - \varphi(t, v)| = |A(t)(u - v)| \leq C'|u - v| \).

A potential drawback of the theorem is that it requires the function \( \varphi(t, u) \) to be globally Lipschitz in \( u \).

What if we only know that
\[ |\varphi(t, u) - \varphi(t, v)| \leq C'|u - v| \quad \forall u, v \in \Omega \]
where \( \Omega \) is some subset of \( \mathbb{R}^n \)?

The problem is that then the solution may blow up and escape \( \Omega \).
In such cases, one can always prove local existence on some interval \([t_0 - S, t_0 + S]\).

Example. Set \( I = [-T, T] \) & \( \Omega = B_R(u_0) = \{ u \in \mathbb{R}^n : |u - u_0| \leq R \} \).

Set \( M = \sup_{(t, u) \in I \times \Omega} |\varphi(t, u)| \) & \( S = \min \left( \frac{R}{M}, T \right) \).

Then \( u \) cannot escape \( \Omega \) in time \( S \) and so we can prove existence & uniqueness on \([-S, S]\).
**Example** \( f(t, u) = u^2 \) & \( u_0 > 0 \)

\[
\begin{cases}
  u = u_0 \\
  u(0) = u_0
\end{cases}
\]

This \( f \) is not globally Lipschitz.

However, for any finite \( A \), it is Lipschitz on \( \Omega = [-A, A] \).

\[ |f(t, u) - f(t, v)| = |u^2 - v^2| = |u - v| \cdot |u + v| \leq 2A |u - v|. \]

We have \( M = \sup_{(A, u) \in \mathbb{R} \times \Omega} f(t, u) = A^2 \)

\[ S = \frac{A \cdot u_0}{M} = \frac{1}{A} \cdot u_0 \]

To maximize \( S \), we set \( A = 2u_0 \), \( \Rightarrow S = \frac{1}{2u_0} - \frac{1}{4u_0} = \frac{1}{4u_0} \)

(The exact solution is \( u(t) = \frac{u_0}{1-u_0 t} \) so we get blow-up at \( t = \frac{1}{u_0} \).)

Sometimes problem-specific information allows us to do better.

**Example** Consider a particle moving in a potential field \( \phi \).

At time \( t \), let \( u(t) \in \mathbb{R}^n \) denote the particle location.

Then at time \( t \), the particle is subjected to the conservative force \( F(u) = -\nabla \phi(u) \).

Newton \( \Rightarrow m \ddot{u}(t) = -\nabla \phi(u) \).

Set \( \rho(t) = m \dot{u}(t) \) \( q(t) = u(t) \) \( \Rightarrow \begin{cases} \ddot{\rho}(t) = m \ddot{u}(t) = -\frac{\partial}{\partial q} \phi(q) \\
\dot{q}(t) = \ddot{u} = \frac{\rho}{m} = \sqrt{\frac{1}{2m} \rho^2} \end{cases} \)

Set \( H(\rho, q) = \frac{1}{2m} \rho^2 + \phi(q) \), then we can write the ODE
\[
\begin{align*}
\dot{p} &= -\nabla q \\
\dot{q} &= \nabla p
\end{align*}
\]
\[
\Leftrightarrow \text{Hamiltonian ODE.}
\]

Note that physically, the Hamiltonian
\[
H(p,q) = \frac{1}{2m}(\dot{p}^2 + \dot{q}^2) = \frac{1}{2} m (\dot{u}(t))^2 + \phi(u(t))
\]
is the total energy.

\[
\frac{d}{dt} H(p,q) = \frac{d}{dt} H(p,q) = \dot{p} \dot{p} + \dot{q} \dot{q} = -q \ddot{q} \cdot \nabla p + \nabla q \cdot \dot{q} H = 0.
\]
So \((p,q)\) stays on the set \[\{(p,q) \in \mathbb{R}^2 : H(p,q) = c_0\} =: \Omega_0\]

If the map \(f(t,[u]) = \left[\begin{array}{c} -q \ddot{q} H \\ \frac{\dot{p}}{m} \end{array}\right] \quad \Omega = \left[\begin{array}{c} -q \phi(q) \\ \frac{\dot{p}}{m} \end{array}\right]\)
is Lipschitz in some neighborhood of \(\Omega_0\), then global existence is assured. (For this, a sufficient condition is that \(\partial^2 \phi\) is bounded.)

More generally, consider
\[
\begin{align*}
\dot{u} &= -\nabla v(u) \\
u(t_0) &= u_0
\end{align*}
\]
Then
\[
\frac{d}{dt} v(u(t)) = \dot{u} \cdot \nabla v(u) = -\nabla v(u) \cdot \nabla v(u) = -|\nabla v(u)|^2 \leq 0
\]
Thus the soln stays inside the set \(\Omega = \{u : v(u) \leq v(u_0)\}\).

If this is a closed set, and if \(f(t,u) = -\nabla v(u)\) is uniformly Lipschitz on \(I \times \Omega\), then existence and uniqueness are assured on \(I\).