Homework set 1 — APPM5440, Fall 2009

Problem 2c: Set $I = [0, 1]$ and consider the set $X$ consisting of all continuous functions on $I$, with the norm

$$||f|| = \int_0^1 |f(x)| \, dx.$$ 

Prove that the space $X$ is not complete.

Solution: A straight-forward way of proving this is to construct a Cauchy-sequence that does not have a limit point in $X$. One example is

$$f_n(x) = \begin{cases} 
-1 & x < 1/2 - 1/n, \\
\frac{1}{n}(x - 1/2) & 1/2 - 1/n \leq x \leq 1/2 + 1/n, \\
1 & x > 1/2 + 1/n.
\end{cases}$$

We first prove that $(f_n)$ is Cauchy. Note that for any $m, n$, and $x$, we have $|f_n(x) - f_m(x)| \leq 1$. When $m, n \geq N$, we further have $f_n(x) - f_m(x) = 0$ outside the interval $[1/2 - 1/N, 1/2 + 1/N]$, so

$$||f_n - f_m|| = \int_{1/2-1/N}^{1/2+1/N} |f_n(x) - f_m(x)| \, dx \leq \int_{1/2-1/N}^{1/2+1/N} 1 \, dx = 2/N.$$ 

We next prove that $(f_n)$ cannot converge to any element in $X$. Pick an arbitrary $\varphi \in X$. Assume temporarily that $\varphi(1/2) \geq 0$. Since $\varphi$ is continuous, there exists a $\delta > 0$ such that $\varphi(x) \geq -1/2$ for $x \in B_\delta(1/2)$. Pick an integer $N > 2/\delta$. Then, for $n \geq N$, we have $f_n(x) = -1$ when $x \in [1/2 - \delta, 1/2 - \delta/2]$, and so

$$||f_n - \varphi|| \geq \int_{1/2-\delta}^{1/2-\delta/2} |f_n(x) - \varphi(x)| \, dx \geq \int_{1/2-\delta}^{1/2-\delta/2} 1/2 \, dx = \delta/4.$$ 

If on the other hand $\varphi(1/2) < 0$, then pick $\delta > 0$ such that $\varphi(x) \leq 1/2$ on $[1/2, 1/2 + \delta]$ and proceed analogously. \hfill \Box

Remark 1: Note that you cannot solve a problem like the one above by constructing a Cauchy sequence $(f_n)$ in $X$, point to a non-continuous function $f$, and claim that since $f_n$ “converges to $f$”, $X$ cannot be complete. Note that the metric is not even defined for functions outside of $X$.

Remark 2: Can you somehow add the limit points of Cauchy sequences in $X$ and obtain a complete space $\bar{X}$? The answer is yes, you can do that for any metric space; the resulting space $\bar{X}$ is called the “completion” of $X$ and is (in a certain sense) unique. For the present example, $\bar{X}$ is the set of all (Lebesgue measurable) real-valued functions on $I$ for which

$$\int_0^1 |f(x)| \, dx < \infty,$$

where the integral is what is called a “Lebesgue” integral. This space is denoted $L^1(I)$. Strictly speaking, an element of $L^1(I)$ is an equivalence class of functions that differ only on a set of Lebesgue measure zero. This roughly means that two functions $f$ and $g$ are considered identical if

$$\int_0^1 |f(x) - g(x)| \, dx = 0.$$