Homework set 5 — APPM5440 — Spring 2009

2.7: Set \( I = [0, 1] \), and \( \Omega = \{ f \in C(I) : \text{Lip}(f) \leq 1, \int f = 0 \} \).

We will use the Arzelà-Ascoli theorem, of course.

The Lipschitz condition implies that \( \Omega \) is equicontinuous. (To prove this, fix any \( \varepsilon > 0 \). Set \( \delta = \varepsilon \). Then for any \( f \in \Omega \), and \( |x - y| < \delta \), we have \( |f(x) - f(y)| \leq \text{Lip}(f) |x - y| \leq |x - y| < \varepsilon \).

To prove that \( \Omega \) is bounded, note that if \( \int f = 0 \), and \( f \) is continuous, then there must exist an \( x_0 \in I \) such that \( f(x_0) = 0 \). Then for any \( x \in I \) and any \( f \in \Omega \), we have \( |f(x)| = |f(x) - f(x_0)| \leq \text{Lip}(f) |x - x_0| \leq |x - x_0| \leq 1 \). So \( ||f||_u \leq 1 \).

Finally we need to prove that \( \Omega \) is closed. Let \( (f_n) \) be a Cauchy sequence in \( \Omega \). Since \( C(I) \) is complete, there exists an \( f \in C(I) \) such that \( f_n \to f \) uniformly. We need to prove that \( f \in \Omega \). Since \( f_n \to f \) uniformly, we know both that \( \text{Lip}(f) \leq \limsup_{n \to \infty} \text{Lip}(f_n) \leq 1 \), and that \( \int f = \lim_{n \to \infty} \int f_n = 0 \). This proves that \( f \in \Omega \).
2.8: We will explicitly construct a dense countable subset \( \Omega \) of \( C([a, b]) \). Without loss of generality, we can assume that \( a = 0 \) and that \( b = 1 \).

For \( n = 1, 2, \ldots \), and for \( j = 0, 1, 2, \ldots, n \), set \( x_j^{(n)} = j/n \). Let \( \Omega_n \) denote the subset of \( C(I) \) of functions that (1) are linear on each interval \([x_{j-1}^{(n)}, x_j^{(n)}]\), and (2) take on rational values for each \( x_j^{(n)} \). Since each function in \( \Omega_n \) is uniquely defined by its values on the \( x_j^{(n)} \)'s, we can identify \( \Omega_n \) by \( \mathbb{Q}^{n+1} \).

Hence \( \Omega_n \) is countable.

Set \( \Omega = \bigcup_{n=1}^{\infty} \Omega_n \). Since each \( \Omega_n \) is countable, \( \Omega \) is countable.

It remains to prove that \( \Omega \) is dense in \( C(I) \). Fix any \( f \in C(I) \), and any \( \varepsilon > 0 \). Since \( I \) is compact, \( f \) is uniformly continuous on \( I \) so there exists a \( \delta > 0 \) such that \( |x - y| < \delta \) implies that \( |f(x) - f(y)| < \varepsilon/2 \). Pick an \( n \) such that \( 1/n < \delta \), and pick a \( \varphi \in \Omega_n \) such that \( |\varphi(x_j^{(n)}) - f(x_j^{(n)})| < \varepsilon/2 \) for \( j = 0, 1, 2, \ldots, n \). We will prove that \( ||\varphi - f||_u < \varepsilon \): Fix an \( x \in I \). Then pick \( j \in \{1, 2, \ldots, n\} \) so that \( x \in [x_{j-1}^{(n)}, x_j^{(n)}] \). Since \( \varphi \) is linear in this interval, there is a number \( \alpha \in [0, 1] \) such that

\[
\varphi(x) = \alpha \varphi(x_{j-1}^{(n)}) + (1 - \alpha) \varphi(x_j^{(n)}).
\]

Now

\[
(1) \quad |f(x) - \varphi(x)| = |\alpha f(x) + (1 - \alpha) f(x) - \alpha \varphi(x_{j-1}^{(n)}) - (1 - \alpha) \varphi(x_j^{(n)})| \\
\leq \alpha |f(x) - \varphi(x_{j-1}^{(n)})| + (1 - \alpha) |f(x) - \varphi(x_j^{(n)})|.
\]

Since \( |f(x) - f(x_{j-1}^{(n)})| \leq \varepsilon/2 \) (by the uniform continuity) and since \( |f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)})| < \varepsilon/2 \) (by the choice of \( \varphi \)), we have

\[
(2) \quad |f(x) - \varphi(x_{j-1}^{(n)})| \leq |f(x) - f(x_{j-1}^{(n)})| + |f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)})| < \varepsilon.
\]

Analogously,

\[
(3) \quad |f(x) - \varphi(x_j^{(n)})| \leq |f(x) - f(x_j^{(n)})| + |f(x_j^{(n)}) - \varphi(x_j^{(n)})| < \varepsilon.
\]

Together, (1), (2), and (3) imply that \( |f(x) - \varphi(x)| < \varepsilon \).
2.9: (a) Suppose that \( w(x) > 0 \) for \( x \in (0, 1) \). Then \(|\cdot|_w\) is a norm since:
(i) \( ||\lambda f||_w = \sup_x w(x) |\lambda f(x)| = |\lambda| \sup_x w(x) |f(x)| = |\lambda| ||f||_w \).
(ii) \( ||f + g||_w = \sup_x w(x) |f(x) + g(x)| \leq \sup_x w(x) (|f(x)| + |g(x)|) \leq \sup_x w(x) |f(x)| + \sup_x w(x) |g(x)| = ||f||_w + ||g||_w \).
(iii) If \( f = 0 \), then clearly \( ||f||_w = 0 \). Conversely, if \( f \neq 0 \), then \( f(x_0) \neq 0 \) for some \( x_0 \in (0, 1) \). Then \( ||f||_w \geq w(x_0) |f(x_0)| > 0 \).

(b) Assume that \( w(x) > 0 \) for \( x \in [0, 1] =: I \). Set \( m = \inf_{x \in I} w(x) \) and \( M = \sup_{x \in I} w(x) \). Since \( I \) is compact and \( w \) is continuous, \( w \) attains both its inf and its sup, and therefore \( m > 0 \) and \( M < \infty \). Then
\[ ||f||_u = \sup_{x \in I} |f(x)| \geq \sup_{x \in I} \frac{w(x)}{M} |f(x)| = \frac{1}{M} ||f||_w. \]
and
\[ ||f||_u = \sup_{x \in I} |f(x)| \leq \sup_{x \in I} \frac{w(x)}{m} |f(x)| = \frac{1}{m} ||f||_w. \]
It follows that
\[ \frac{1}{M} ||f||_w \leq ||f||_u \leq \frac{1}{m} ||f||_w. \]

(c) Set \( ||f|| = \sup_{x \in I} |xf(x)| \). We will prove that \( ||| \cdot ||| \) is not equivalent to the uniform norm. Set for \( n = 1, 2, \ldots \)
\[ f_n(x) = \begin{cases} 1 - nx & x \in [0, 1/n], \\ 0 & x \in (1/n, 1]. \end{cases} \]
Then \( ||f_n||_u = 1 \) for all \( n \), while \( ||f_n|| = \sup_{x \in I} x |f_n(x)| \leq 1/n \). This proves that there cannot be a finite \( M \) such that \( ||f||_u \leq M ||f||_w \) for all \( f \).

(d) We will prove that the set \( C(I) \) equipped with the norm \( ||| \cdot ||| \) is not a Banach space by constructing a Cauchy sequence with no limit point in \( C(I) \). For \( n = 1, 2, \ldots \), define \( f_n \in C(I) \) by
\[ f_n(x) = \begin{cases} x^{-1/2} & x \in (1/n, 1], \\ \sqrt{n} & x \in [0, 1/n]. \end{cases} \]
Fix a positive integer \( N \). Then, if \( m, n \geq N \), we have
\[ |||f_n - f_m||| = \sup_{x \in [0,1/N]} x |f_n(x) - f_m(x)| \leq \sup_{x \in [0,1/N]} (x|f_n(x)| + x|f_m(x)|) \leq \sup_{x \in (0,1/N)} (x \cdot x^{-1/2} + x \cdot x^{-1/2}) = 2N^{-1/2}. \]
Consequently, \( (f_n)_{n=1}^\infty \) is a Cauchy sequence. But \( f_n \) cannot converge uniformly to any function in \( C(I) \). (To prove the last contention, suppose that \( f_n \to f \) for some \( f \in C(I) \). Then \( f(0) = \lim_{n \to \infty} f_n(0) = \infty \), which is a contradiction.)