Problem 1: In what follows, $(X, d_X)$ and $(Y, d_Y)$ are metric spaces.

(a) Define what it means for a function $f : X \to Y$ to be continuous.

(b) Define what it means for a subset $\Omega$ of $X$ to be compact.

(c) Define what a completion of $(X, d_X)$ is.

(d) Let $\Omega$ be a subset of $X$. Define the closure of $\Omega$.

Solution:

(a) For instance: $f$ is continuous if $f^{-1}(G)$ is open whenever $G$ is open.

(b) $\Omega$ is compact if every open cover has a finite subcover.

Many provided the answer that $\Omega$ is compact if it is totally bounded and complete. One could define compactness this way, so the answer gave full credit, but it is non-standard.

(c) A metric space $(Y, d_Y)$ is a completion of $(X, d_X)$ if:

1. There is an isometry $i : X \to Y$.
2. $i(X)$ is dense in $Y$.
3. $(Y, d_Y)$ is complete.

Answering that you “add the limit points” is too imprecise, and does not result in full credit.

(d) The closure is the intersection of all closed sets that contain $\Omega$. Alternatively, you could define the closure as the set of all limit points of sequences in $\Omega$.

The answer “the smallest closed set that contains $\Omega$” gave full credit but is not great since you would then have to prove that such a smallest set in fact exists.
Problem 2: Let $\mathbb{Q}$ denote the set of rational numbers. On $\mathbb{Q}$, we define the discrete metric

$$d(x, y) = \begin{cases} 
0, & \text{when } x = y, \\
1, & \text{when } x \neq y.
\end{cases}$$

(a) What subsets of $\mathbb{Q}$ are open in $(\mathbb{Q}, d)$? Prove your claim.

(b) Specify which sequences in $(\mathbb{Q}, d)$ are convergent. No motivation required.

(c) Set $\Omega = \{ q \in \mathbb{Q} : q^2 < 2 \}$. What is the closure of $\Omega$ in $(\mathbb{Q}, d)$? No motivation required.

(d) Set $\Omega = \{ q \in \mathbb{Q} : q^2 < 2 \}$. What is the completion of $(\Omega, d)$? No motivation required.

Solution:

(a) All subsets are open. To prove this, let $\Omega$ be an arbitrary subset of $\Omega$. Let $x \in \Omega$. We need to prove that for some $\varepsilon$, there is an $\varepsilon$-ball centered at $x$ that is entirely contained in $\Omega$. Set $\varepsilon = 1/2$. Then $B_\varepsilon(x) = \{ x \}$ which is clearly contained in $\Omega$.

(b) A sequence $(x_n)_{n=1}^\infty$ is convergent iff it is constant beyond a certain point. In other words:

$$(x_n)_{n=1}^\infty \text{ is convergent } \iff \exists N \text{ such that } x_n = x_N \text{ when } n \geq N.$$  

The “$\iff$” should be obvious. To prove “$\Rightarrow$”, suppose that $(x_n)$ is Cauchy. Then there exists $N$ such that $d(x_n, x_N) < 1/2$ when $n \geq N$. Note that if $d(x_n, x_N) < 1/2$, then $x_n = x_N$.

(c) Since every subset in $(\mathbb{Q}, d)$ is open, every subset is also closed. Therefore, $\overline{\Omega} = \Omega$.

(d) Since only sequences that become constant are Cauchy, $(\Omega, d)$ is complete in its own right. Therefore, the completion of $(\Omega, d)$ is $(\Omega, d)$. 
Problem 3: Let \((x_n)_{n=1}^{\infty}\) be a sequence of real numbers. Set \(y_n = x_{2n}\). Which of the following two statements must necessarily be true:

(a) \(\limsup_{n \to \infty} x_n \leq \limsup_{n \to \infty} y_n\),

(b) \(\limsup_{n \to \infty} y_n \leq \limsup_{n \to \infty} x_n\).

Motivate your answers carefully. State the definition of “limsup” that you use and make sure that your argument follows directly from this definition.

Solution: Definition of “limsup”:

\[
\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \left( \sup_{k \geq n} \{x_k\} \right)
\]

(a) This is not necessarily true. For a counterexample, set \(x_n = (-1)^{n+1}\) so that \(x_n = 1\) if \(n\) is odd and \(x_n = -1\) if \(n\) is even. We have

\[
\limsup_{n \to \infty} x_n = 1.
\]

On the other hand, \(y_n = x_{2n} = (-1)^{2n+1} = -1\), so

\[
\limsup_{n \to \infty} y_n = -1.
\]

(b) This is true. We have

\[
\limsup_{n \to \infty} y_n = \lim_{n \to \infty} \sup_{k \geq n} \{x_{2k} : k \geq n\} \leq \lim_{n \to \infty} \sup_{k \geq n} \{x_{2k}, x_{2k+1} : k \geq n\} = \lim_{n \to \infty} \sup_{k \geq 2n} \{x_k\} = \limsup_{n \to \infty} x_n.
\]

The inequality in the second step holds since we enlarge the set over which the sup is taken.
**Problem 4:** Let \((X, || \cdot ||)\) be a normed linear space. Suppose that every sequence \((x_n)_{n=1}^{\infty}\) in \(X\) such that \(||x_m - x_n|| \leq 1/N\) whenever \(m, n \geq N\) converges to a point in \(X\). Is \((X, || \cdot ||)\) necessarily complete? Prove this if you answer yes, and give a counterexample if you answer no.

**Solution:** Yes, the set must be complete, as we will prove.

Let \((y_n)_{n=1}^{\infty}\) be an arbitrary Cauchy sequence.

From \((y_n)\), we pick a rapidly convergent subsequence as follows:

- Pick \(n_1\) such that \(m, n \geq n_1 \implies ||y_m - y_n|| \leq 1\).
- Pick for \(j = 2, 3, \ldots\) an integer \(n_j > n_{j-1}\) such that \(m, n \geq n_j \implies ||y_m - y_n|| \leq 1/j\).

The sequence \((y_{n_j})_{j=1}^{\infty}\) satisfies \(i, j \geq J \implies ||y_{n_i} - y_{n_j}|| \leq 1/J\), so our assumption on \((X, || \cdot ||)\) implies that there exists a point \(y \in X\) such that \(y_{n_j} \to y\) as \(j \to \infty\).

It only remains to prove that \(y_n \to y\) as \(n \to \infty\). Pick \(\epsilon > 0\). Since \((y_n)_{n=1}^{\infty}\) is Cauchy, there exists an \(N\) such that

\[ m, n \geq N \implies ||y_m - y_n|| < \epsilon/2. \]

Now pick \(j_0\) such that \(n_{j_0} \geq N\) and \(d(y_{n_{j_0}}, y) < \epsilon/2\). Then if \(n \geq N\), we have

\[ d(y_n, y) \leq d(y_n, y_{n_{j_0}}) + d(y_{n_{j_0}}, y) \leq \epsilon/2 + \epsilon/2 = \epsilon. \]
**Problem 5:** Let \((X, d)\) be a compact metric space. Let \(C_b(X)\) denote the set of all bounded real-valued continuous functions on \(X\), equipped with the uniform norm,

\[
||f||_u = \sup_{x \in X} |f(x)|.
\]

Prove that \(C_b(X)\) is complete.

**Solution:** The assumption that \(X\) is compact is a red herring — this property is not required for the statement to be true.

Let \((f_n)_{n=1}^\infty\) be a Cauchy sequence in \((X, d)\). We will construct a limit function, and then prove that it is bounded, that it is indeed the limit of the sequence in the uniform norm, and finally that it is continuous.

**Step 1 — construct the limit point \(f\):** Fix \(x \in X\). Since \(|f_n(x) - f_m(x)| \leq ||f_n - f_m||\) and \((f_n)_{n=1}^\infty\) is Cauchy, the sequence \((f_n(x))_{n=1}^\infty\) is Cauchy in \(\mathbb{R}\). Since \(\mathbb{R}\) is complete, the sequence is convergent, we can therefore define a function \(f\) via

\[
f(x) = \lim_{n \to \infty} f_n(x).
\]

**Step 2 — prove that \(f\) is bounded:** We have

\[
\sup_{x \in X} |f(x)| = \sup_{x \in X} \left( \lim_{n \to \infty} |f_n(x)| \right) \leq \liminf_{n \to \infty} \left( \sup_{x \in X} |f_n(x)| \right) = \liminf_n ||f_n|| < \infty,
\]

where in the last step we used that \((f_n)\) is Cauchy, and therefore bounded.

**Step 3 — prove that \(f_n \to f\) uniformly:** Fix \(\varepsilon > 0\). Pick \(N\) such that \(||f_m - f_n|| < \varepsilon/2\) when \(m, n \geq N\). Then for \(n \geq N\), we have

\[
||f_n - f|| = \sup_{x \in X} |f_n(x) - f(x)| = \sup_{x \in X} \left( \lim_{m \to \infty} |f_n(x) - f_m(x)| \right)
\]

\[
\leq \liminf_{m \to \infty} \left( \sup_{x \in X} |f_n(x) - f_m(x)| \right) = \liminf_{m \to \infty} ||f_n - f_m|| \leq \varepsilon/2 < \varepsilon.
\]

**Step 4 — prove that \(f\) is continuous:** This follows directly from the fact that each \(f_n\) is continuous and \(f_n \to f\) uniformly (since uniform convergence preserves continuity).

Steps 2 and 4 prove that \(f \in C_b(X)\), and step 3 proves that \(f\) is the limit point of \((f_n)\). The proof is therefore complete.