Problem 1: (24 points) For each of the statements below, state whether it is TRUE or FALSE. (“TRUE” of course means “necessarily true.”) No motivation required.

(a) Define for \( n = 1, 2, 3, \ldots \) the function \( f_n : \mathbb{R} \to \mathbb{R} : x \mapsto e^{-(x-n)^2} \). The sequence \( (f_n)_{n=1}^\infty \) converges pointwise to zero.

(b) With \( f_n \) defined as in (a), the sequence \( (f_n)_{n=1}^\infty \) converges uniformly to zero.

(c) With \( f_n \) defined as in (a), the set \( \Omega = \{ f_n \}_{n=1}^\infty \) is equicontinuous.

(d) With \( f_n \) defined as in (a), the set \( \{ f_n \}_{n=1}^\infty \) is pre-compact in \( C_b(\mathbb{R}) \).

(e) Let \( (g_n)_{n=1}^\infty \) be a sequence of real-valued functions on the set \( I = [0, 1] \) that converges pointwise to a function \( g \). Suppose further that the set \( \{ g_n \}_{n=1}^\infty \) is equicontinuous. Then \( g \) is continuous.

(f) Suppose that \( (h_n)_{n=1}^\infty \) is a sequence of functions \( h_n : \mathbb{R} \to \mathbb{R} \) that converges uniformly to zero. Then \( \int_{-\infty}^{\infty} h_n(t) \, dt \to 0 \).

(g) The set of continuously differentiable functions on the interval \( I = [0, 1] \) is (topologically) closed in \( C_b(I) \).

(h) The set \( \Omega = \{ f \in C_b(I) : ||f||_u \leq 2 : \text{Lip}(f) \leq 3 \} \) is compact in \( C_b(I) \).

Problem 2: (26 points) Set \( A = \{ f \in C_b(\mathbb{R}) : \lim_{t \to \infty} |f(t)| = \lim_{t \to -\infty} |f(t)| = 0 \} \).

(a) Prove that \( A \) is closed in \( C_b(\mathbb{R}) \).

(b) Prove that \( A \) is the closure of the set of compactly supported functions in \( C_b(\mathbb{R}) \).

(c) Is the set \( A \) equipped with the uniform norm a Banach space? Motivate your answer briefly.

(d) Set \( B = \{ f \in C_b(\mathbb{R}) : \sup_{t \in \mathbb{R}} e^{|t|} |f(t)| < \infty \} \). Prove that \( B \) is not closed in \( C_b(\mathbb{R}) \).

Problem 3: (25 points) State the Arzelà-Ascoli theorem. (No proof necessary.) Set \( I = [0, 1] \) and let \( k : I^2 \to \mathbb{R} \) be continuous. Define on \( C_b(I) \) the integral operator

\[
[A u](x) = \int_0^1 k(x, y) u(y) \, dy.

\]

Let \( (u_n)_{n=1}^\infty \) be a bounded sequence in \( C_b(I) \). Prove that \( (Au_n)_{n=1}^\infty \) has a uniformly convergent subsequence.

Problem 4: (25 points) State and prove the contraction mapping theorem.