The Linear Space $B(X,Y)$

Let $X, Y$ be NLS's.

Then $B(X,Y)$ is the set of all bdd linear maps $X \to Y$.

We equip $B(X,Y)$ with the norm

$$
\| T \| = \sup_{x \neq 0} \frac{\| Tx \|_Y}{\| x \|_X} = \sup_{\| x \|_X = 1} \| Tx \|_Y.
$$

Then $B(X,Y)$ is a NLS in its own right.

**Thm** Let $X$ be a NLS & let $Y$ be a Banach space.

Then $B(X,Y)$ is a Banach space.

**Proof** We need to prove completeness.

Let $(T_n)_{n=1}^\infty$ be a Cauchy seq.

* First we construct a putative limit point:
  
  Given $x \in X$, the seq $(T_n x)_{n=1}^\infty$ is Cauchy

  since

  $$
  \| T_n x - T_m x \| \leq \| T_n - T_m \| \| x \|.
  $$

  Since $Y$ is complete, $(T_n x)$ has a limit point,

  define $T x$ as this limit point.

* Is $T$ linear?

  Fix $a, b \in \mathbb{R}$ and $x, y \in X$.

  Then

  $$
  T(ax + by) = \lim_{n \to \infty} T_n (ax + by) = \lim_{n \to \infty} [a T_n x + b T_n y] = a T x + b T y \ \text{Yes}!
  $$

* Is $T$ bdd?

  Since $(T_n)$ is Cauchy, we have $C = \lim_{n \to \infty} \| T_n \|_Y$.

  Then

  $$
  \| T x \| = \lim_{n} \| T_n x \|_Y \leq \limsup_{n} \| T_n \| \| x \| = C \| x \|.
  $$

  Yes.
It remains to prove that \( \|T_n - T\| \to 0 \) as \( n \to \infty \).

Fix \( \varepsilon > 0 \). Since \( (T_n) \) is Cauchy, \( \exists N \) s.t. \( m, n > N \Rightarrow \|T_m - T_n\| < \varepsilon \).

Fix \( x \) s.t. \( \|x\| = 1 \). Then if \( m, n > N \),
\[
\| (T_m - T_n) x \| \leq \| T_m - T_n \| \|x\| < \varepsilon.
\]
(1)

Take limit as \( m \to \infty \) in (1):
\[
\| (T - T_n) x \| = \lim_{m \to \infty} \| T_m x - T_n x \| \leq \varepsilon
\]
(2)

Take sup over \( x \) in (2):
\[
\| T - T_n \| = \sup_{\|x\| = 1} \| (T - T_n) x \| \leq \varepsilon.
\]

Defn: A linear space \( L \) is called an algebra if there is a multiplication operator \( L \times L \to L \) that satisfies
\[
(1) \quad (xy)z = x(yz) \quad \forall x, y, z \in L
\]
\[
(2) \quad x(y+z) = xy + xz \quad \forall x, y, z \in L
\]
\[
(3) \quad a(xy) = (ax)y = x(ay) \quad \forall a \in \mathbb{R}, x, y \in L
\]

A Banach algebra is an algebra that is also a Banach space with a multiplication satisfying
\[
(4) \quad \text{There is an element } e \in L \text{ s.t. } xe = ex = x \quad \forall x \in L
\]
\[
(5) \quad \|xy\| \leq \|x\| \|y\| \quad \forall x, y \in L
\]
Examples of algebras:

* \( \mathcal{C} \) and \( \mathcal{R} \) themselves
* \( \mathcal{C}_b(\mathcal{X}) \) for any topological space \( \mathcal{X} \).
  \( \mathcal{C}_b(\mathcal{X}) \) is in fact a Banach algebra.
* Let \( \Omega \) be a domain in \( \mathcal{C} \).
  Then \( \mathcal{A}(\Omega) \) is space of analytic functions on \( \Omega \).
  \( \mathcal{B}(\Omega) \) is a Banach algebra (with the uniform norm).
* If \( \mathcal{X} \) is a NLS, then \( \mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X}, \mathcal{X}) \) is a C-algebra.
  If \( \mathcal{X} \) is a Banach space, then \( \mathcal{B}(\mathcal{X}) \) is a Banach algebra.

Note: \( \mathcal{B}(\mathcal{X}) \) is almost never commutative!

Def: Suppose that \( T_n, T \in \mathcal{B}(\mathcal{X}, \mathcal{X}) \).
We say that \( T_n \to T \) in norm if \( \| T_n - T \| \to 0 \) as \( n \to \infty \).
We say that \( T_n \to T \) strongly if \( \| T_n x - T x \| \to 0 \) as \( n \to \infty \) for \( x \).

Note: Norm convergence implies strong convergence but the converse is not necessarily true.

Example: \( \mathcal{X} = l^2 \) (with standard norm)
\( P_n : x \mapsto (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \)
\( P_n \to I \) strongly since for any \( x \) we have \( P_n x \to x \).
\( P_n \to I \) in norm since \( \| P_n - I \| = 1 \) for all \( n \).
Example: Let $X$ be a Banach space and suppose that $A \in \mathcal{B}(X)$. 

Set $T_n = \sum_{j=0}^{n-1} \frac{1}{j!} A^j$, then $(T_n)$ is Cauchy since 

$$\|T_n - T_m\| = \left\| \sum_{j=m+1}^{n} \frac{1}{j!} A^j \right\| \leq \sum_{j=m+1}^{n} \frac{\|A^j\|}{j!} \to 0 \quad \text{as} \quad m, n \to \infty.$$ 

Thus $T_n$ converges in norm to an element $\exp(A)$ 

$$\exp(A) = \sum_{j=0}^{\infty} \frac{1}{j!} A^j, \quad \|\exp(A)\| \leq e^{\|A\|}.$$ 

If $X = \mathbb{R}^n$ and $A = VDV^{-1}$, then $\exp(A) = V \exp(D) V^{-1}$. 

If $A$ and $B$ commute, then $\exp(A+B) = \exp(A) \exp(B)$. 

Now consider the ODE 

$$\begin{cases} 
\frac{dx}{dt} = X(x(t)) \\
x(0) = x_0 
\end{cases}$$ 

where $\frac{dx}{dt}$ is defined as the uniform limit $\lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$. 

The solution is $x(t) = \exp(A t) x_0$. 

Proof: 

$$\frac{1}{n} \left( x(\frac{t}{n}) - x(0) \right) - A x(0) = \frac{1}{n} \left( e^{A \frac{t}{n}} x_0 - e^{A t} x_0 \right) - A e^{A t} x_0 = \frac{1}{n} \left( e^{A t} - 1 \right) x_0 = \frac{1}{n} \left( e^{A t} \sum_{n=1}^{\infty} \frac{(-A)^n}{n!} \right) x_0 = e^{A \frac{t}{n}} \left( \sum_{n=2}^{\infty} \frac{(-A)^n}{n!} \right) x_0 \to 0 \quad \text{as} \quad n \to \infty.$$ 

$$\|x(t) - x(0)\| = \left\| \exp(A t) x_0 \right\| \leq e^{\|A\| t} \|x_0\| \to 0 \quad \text{as} \quad t \to \infty.$$