Problem 1: Consider the set $\mathbb{R}^n$ equipped with the norm 

$$ ||x||_p = \left( \sum_{j=1}^{n} |x_j|^p \right)^{1/p} $$

(a) Prove that $|| \cdot ||_p$ is a norm for $p = 1$.

(b) Prove that $|| \cdot ||_p$ is a norm for $p = 2$.

(c) Prove that $\lim_{p \to \infty} ||x||_p = \max_{1 \leq j \leq n} |x_j|$.

(d*) Prove that $|| \cdot ||_p$ is a norm for $p \in (1, \infty)$. (See hint on next page.)

(e*) For $x, y \in \mathbb{R}^n$, let $d_{\text{hamming}}(x, y)$ denote the number of non-zero entries of $x - y$. Is $d_{\text{hamming}}$ a metric on $\mathbb{R}^n$? Prove that $d_{\text{hamming}}(x, y) = \lim_{p \to 0} ||x - y||_p^p$.

Problem 2: Set $I = [0, 1]$ and consider the set $X$ consisting of all continuous functions on $I$. Define an addition and a scalar multiplication operator that make $X$ a normed linear space.

(a) Which of the following candidates define a norm on $X$:

- $||f||_a = \sup_{0 \leq x \leq 1} |f(x)|$
- $||f||_b = \sup_{0 \leq x \leq 1/2} |f(x)|$
- $||f||_c = \sup_{0 \leq x \leq 1} |f(x)|^2$
- $||f||_d = 2 \sup_{0 \leq x \leq 1} |f(x)|$
- $||f||_e = \sup_{0 \leq x \leq 1} (1 + \cos x)|f(x)|$
- $||f||_f = |f(0)| + \sup_{0 \leq x \leq 1} |f(x)|$
- $||f||_g = |f(0)|$

(b) Prove that 

$$ ||f|| = \int_0^1 |f(x)| \, dx $$

is a norm on $X$.

(c) Prove that with respect to the norm given in (b), the space $X$ is not complete.
Hint for 1d:

You may want to use the Hölder inequality: Let $p$ and $q$ be numbers in the interval $(1, \infty)$ such that $1/p + 1/q = 1$, and let $(\alpha_j)_{j=1}^n$ and $(\beta_j)_{j=1}^n$ be vectors in $\mathbb{R}^n$. Then

$$\sum_{j=1}^n |\alpha_j \beta_j| \leq \left( \sum_{j=1}^n |\alpha_j|^p \right)^{1/p} \left( \sum_{j=1}^n |\beta_j|^q \right)^{1/q}.$$  

(You can look up a proof in, e.g., Wikipedia. You will also see that the inequality is far more general than what is stated here.)

Next let $x, y$ be two non-zero vectors and let $r \in (1, \infty)$. Then

$$||x + y||_r = \sum |x_j + y_j|^r = \sum |x_j + y_j|^{r-1} |x_j + y_j| \leq \sum |x_j + y_j|^{r-1} (|x_j| + |y_j|).$$

Now use the Hölder inequality for suitably chosen $p$ and $q$. 