Homework set 2 — APPM5440, Fall 2012

From the textbook: 1.8, 1.9, 1.10 (important), 1.12, 1.13.

In problem 1.12, it would make a good exercise to prove the statement three times, using each of the three different definitions of continuity defined in class.

Optional: 1.7.

Problem 1: Let \((X, d)\) be a metric space, and let \(\Omega \subseteq X\). Prove that:

\[
\Omega \text{ is dense in } X \iff \forall x \in X, \varepsilon > 0, \exists y \in \Omega \text{ such that } x \in B_\varepsilon(y).
\]

(In words: \(\Omega\) is dense iff for every \(x\) in \(X\), and for every \(\varepsilon > 0\), there exists an element \(y \in \Omega\) that is within distance \(\varepsilon\) of \(x\).

Problem 2: Suppose that \((x_n)_{n=1}^\infty\) and \((y_n)_{n=1}^\infty\) are Cauchy sequences in a metric space \((X, d)\). Prove that the sequence \((d(x_n, y_n))_{n=1}^\infty\) converges.

Problem 3*: The proof that every metric space has a completion that we postponed contains an important technique called the Cantor diagonal argument. Try to use it to prove that the real numbers are not countable. Hint: Assume that there exists an enumeration \((r^{(n)})_{n=1}^\infty\) of all real numbers in the interval \((0, 1)\). Suppose that each \(r^{(n)}\) has a binary number expansion

\[
r^{(n)} = 0.b_1^{(n)} b_2^{(n)} b_3^{(n)} \ldots
\]

(so that each \(b_j^{(n)}\) is either 0 or 1) and use the “diagonal” technique to construct a real number that is not in the sequence. (There is a full solution in the Wikipedia article on Cantor’s diagonal argument.)