Homework set 8 — APPM5440 — Fall 2012

From the textbook: 3.6, 3.7.

On the next page, you’ll find the 2005 midterm. Problem 4 on that midterm is part of this week’s homework. (Note that the questions on topological spaces are outside the syllabus for the midterm this year.)

Problem 1: Let $X$ be a set with infinitely many members. We define a collection $\mathcal{T}$ of subsets of $X$ by saying that a set $\Omega \in \mathcal{T}$ if either $\Omega^c = X \setminus \Omega$ is finite, or if $\Omega$ is the empty set. Verify that $\mathcal{T}$ is a topology on $X$. This topology is called the “co-finite” topology on $X$. Describe the closed sets.

Problem 2: Let $X$ denote a finite set, and let $\mathcal{T}$ be a metrizable topology on $X$. Prove that $\mathcal{T}$ is the discrete topology on $X$.

Problem 3: Consider the set $X = \{a, b, c\}$, and the collection of subsets $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Is $\mathcal{T}$ a topology? Is $\mathcal{T}$ a metrizable topology?
Problem 1: State the Arzelà-Ascoli theorem.

Problem 2: We describe two mathematical objects below. For each description, either provide an example of such an object, or explain why it does not exist. (Brief answers, please!)

(1) A collection of functions in $C(\mathbb{R})$ that is equicontinuous but not uniformly equicontinuous.

(2) A collection of functions in $C([0, 1])$ that is equicontinuous but not uniformly equicontinuous.

Problem 3: Let $X = \{a, b, c, d\}$ be a set, and let

$S = \{\emptyset, X, \{b, d\}, \{a, c\}, \{d\}, \{a, b, c\}\}$

be a collection of subsets of $X$.

(1) Prove that $S$ is not a topology on $X$.

(2) Let $\mathcal{T}$ denote the smallest topology on $X$ that contains $S$ (in other words, $\mathcal{T}$ is the topology generated by the sub-base $S$). List the sets that are contained in $\mathcal{T}$ but not in $S$.

Problem 4: Consider the integral equation

(*)

$u(x) = \pi^2 \sin(x) + \frac{3}{2} \int_0^{\cos(x)} |x - y| u(y) dy.$

Prove that (*) has a unique solution in $C([0, 1])$.

Problem 5: Let $X$ be a topological space, let $Y$ be a Hausdorff space, and let $f$ and $g$ be continuous maps from $X$ to $Y$. Is it necessarily the case that the set $\Omega = \{x \in X : f(x) = g(x)\}$ is closed? Justify your answer by either giving a proof or a counter example.