Problem 1: (24 points) The following questions are worth 8 points each.

(a) Specify which of the following could potentially be the set \( C \) of cluster points of a sequence \( (x_n)_{n=1}^{\infty} \) of real numbers. Any negative answer needs a brief motivation.

1. \( C = [0, 1] \).
2. \( C = (0, 1) \).
3. \( C = [0, \infty) \).
4. \( C = \mathbb{Q} \) (the set of rational numbers).

(Recall that given a sequence \( (x_n) \), its set of cluster points is defined as the set of limit points of sub-sequences of \( (x_n) \).)

(b) Let \( (X; d) \) be a metric space. State the definition of the completion of \( (X; d) \).

(c) Which of the following statements are true (no motivations required):

1. If \( (x_n)_{n=1}^{\infty} \) is a sequence of real numbers, then \( \limsup_{n \to \infty} x_n \) exists.
2. If \( (X; d) \) is a compact metric space, and \( (x_n)_{n=1}^{\infty} \) is a sequence in \( X \) with the property that every convergent subsequence has the same limit \( x \), then \( x_n \to x \).
3. Every compact subset of a metric space is necessarily closed.
4. If \( (X; d) \) is a compact metric space and \( f : X \to (0, 1) \) is continuous, then the function \( g(x) = 1/(1 - f(x)) \) is bounded on \( X \).
5. Let \( X \) be a normed linear space, and the \( B \) denote the unit ball around the origin. Then \( B \) is necessarily totally bounded.

Solution:

(a) Only the set in (1) is possible. (Recall from the homework that \( C \) must be closed. The set \( [0, \infty) \) is a little bit tricky since it is not closed in the set of extended real numbers. So while \( C = [0, \infty] \) is possible, \( C = [0, \infty) \) is not. This subquestion was graded generously. Note that a set being infinite or uncountable is unproblematic. Consider, e.g., the case where \( (x_n) \) is an enumeration of the positive rational numbers. Then \( C = [0, \infty] \).)

(b) See text.

(c) (1) is true.

(2) is true (see homework).

(3) is true.

(4) is true since a continuous function on a compact set attains its max.
(Alternatively, note that \( f(X) \) must be a closed subset of \( (0, 1) \).)

(5) False.
Problem 2: (24 points) Suppose that $(X_1, d_1)$, $(X_2, d_2)$, and $(X_3, d_3)$ are metric spaces, and that $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are continuous. Prove that the composition $h = g \circ f$ defined by

$$h : X_1 \to X_3 : x \mapsto g(f(x))$$

is continuous. State explicitly which definition of continuity you use in your proof.

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Solution:

Definition: A function $f$ is continuous if the pre-image of any open set is open.

Let $G$ be an open subset of $X_3$.

Since $g$ is continuous, $g^{-1}(G)$ is open in $X_2$.

Since $f$ is continuous, $f^{-1}(g^{-1}(G))$ is open in $X_1$.

Since $f^{-1}(g^{-1}(G)) = h^{-1}(G)$, this shows that $h$ is continuous.
Problem 3: (24 points) Set $I = [-1, 1]$ and let $X$ denote the set of real valued continuous functions on $I$. For $f \in X$, define the norm
\[ ||f|| = \int_{-1}^{1} |f(x)| \, dx. \]
Show that $X$ is not a Banach space with respect to this norm.

Solution:
Define $f_n \in X$ via
\[ f_n(x) = \begin{cases} 
-1 & -1 \leq x < -1/n \\
mx & -1/n \leq x < 1/n \\
1 & 1/n \leq x < 1 
\end{cases} \]
The sequence $(f_n)$ is Cauchy. To prove this, suppose that $N \leq m \neq n$. Then
\[ ||f_n - f_m|| = \int_{-1}^{1} |f_n(x) - f_m(x)| \, dx = \int_{-1/N}^{1/N} |f_n(x) - f_m(x)| \, dx \leq \int_{-1/N}^{1/N} dx \leq 2/N. \]
It remains to show that $(f_n)$ cannot converge to any function $g \in X$. Fix $g \in X$. Then
\[ ||f_n - g|| = \int_{-1}^{1} |f_n(x) - g(x)| \, dx = A_n + B_n + C_n \]
where
\[ A_n = \int_{-1}^{-1/n} |g(x) + 1| \, dx, \quad B_n = \int_{-1/n}^{1/n} |f_n(x) - g(x)| \, dx, \quad C_n = \int_{1/n}^{1} |g(x) - 1| \, dx, \]
Set $M = \sup_{x \in I} |g(x)|$. Note that $M$ is finite since $I$ is compact. Since $|f_n(x) - g(x)| \leq 1 + M$ it then follows that $B_n \leq 2M/n$ and so $\lim_{n \to \infty} B_n = 0$. Then
\[ \lim_{n \to \infty} ||f_n - g|| = \int_{-1}^{0} |g(x) + 1| \, dx + \int_{0}^{1} |g(x) - 1| \, dx. \]
Since $g$ is continuous, at least one of the two terms must be non-zero. \(^1\)

\(^1\)For full marks on this problem, the last assertion did not need to get proven. But for the curious, note that this follows from elementary analysis. Suppose $g(0) \neq 1$. Set $\varepsilon = |1 - g(0)|/2$. Then pick $\delta > 0$ such that $|g(x) - g(0)| < \varepsilon$ for $|x| \leq \delta$. Then $\int_{0}^{\delta} |g(x) - 1| \, dx \geq \int_{0}^{\delta} |g(x) - g(0)| \, dx = \delta \varepsilon > 0$. If $g(0) = 1$, then you can analogously prove that $\int_{-\delta}^{0} |g(x) + 1| \, dx > 0$. 
**Problem 4:** (28 points) Let $X$ denote the set of sequences of real numbers $x = (x_1, x_2, x_3, \ldots)$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$, and define for $x \in X$ the norm $||x|| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$. Consider the following four subsets of $X$:

- Let $d$ be a positive integer $d$ and set $A = \{x = (x_1, x_2, \ldots, x_d, 0, 0, \ldots) : \sum_{n=1}^{d} x_n^2 \leq 1\}$.
- $B = \{x = (x_1, x_2, x_3, \ldots) : \sum_{n=1}^{\infty} n^2 x_n^2 \leq 1\}$.
- $C = \{x = (x_1, x_2, x_3, \ldots) : \sum_{n=1}^{\infty} x_n^2 \leq 1\}$.
- $D = \{x = (x_1, x_2, x_3, \ldots) : \sum_{n=1}^{\infty} |x_n| = 1\}$.

Which of the sets $A$, $B$, $C$, and $D$ are compact?

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**Solution:**

Set $A$ is compact. It is isomorphic with the closed unit ball in $\mathbb{R}^d$, which is compact by the Heine-Borel theorem.

Set $B$ is compact. We first prove that $B$ is totally bounded. Pick $\varepsilon > 0$. Pick $N$ such that $N > 2/\varepsilon$. Then for $x \in X$, set

$$P_N x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots).$$

For $x \in B$, we then have $||x - P_N x||^2 = \sum_{n=N+1}^{\infty} x_n^2 \leq \frac{1}{N^2} \sum_{n=N+1}^{\infty} n^2 x_n^2 \leq \frac{1}{N^2} \leq \frac{\varepsilon^2}{4}$.

Observe that $P_N B$ is compact by the Heine-Borel theorem. Let $\{B_{\varepsilon/2}(x^{(j)})\}_{j=1}^{N}$ denote a finite $\varepsilon/2$-cover of $P_N B$. Then for any $x \in B$, we know that $||x^{(j)} - P_N x|| < \varepsilon/2$ for some $x^{(j)}$, and then $||x - x^{(j)}|| \leq ||x - P_N x|| + ||P_N x - x^{(j)}|| < \varepsilon/2 + \varepsilon/2$.

Therefore $\{B_{\varepsilon}(x^{(j)})\}_{j=1}^{N}$ is an $\varepsilon$-cover of $B$.

Next we show that $B$ is closed. Suppose $(x^{(j)})_{j=1}^{\infty}$ is a Cauchy sequence in $B$. Since $X$ is complete, there is an $x \in X$ such that $x^{(j)} \to x$. This in particular implies that $\lim_{j \to \infty} x^{(j)}_n = x_n$ for every $j$. We then find

$$\sum_{n=1}^{\infty} n^2 x_n^2 = \sup_{N} \lim_{j \to \infty} \sum_{n=1}^{N} n^2 (x^{(j)}_n)^2 \leq \lim_{j \to \infty} \sup_{N} \sum_{n=1}^{N} n^2 (x^{(j)}_n)^2 \leq \lim_{j \to \infty} \sum_{n=1}^{\infty} n^2 (x^{(j)}_n)^2 \leq 1.$$

$C$ is not compact. Consider the vectors

$$e^{(1)} = (1, 0, 0, \ldots), \quad e^{(2)} = (0, 1, 0, 0, \ldots), \quad e^{(3)} = (0, 0, 1, 0, 0, \ldots).$$

We find that $e^{(j)} \in C$ for every $j$. Since $||e^{(j)} - e^{(k)}|| = \sqrt{2}$ whenever $j \neq k$, the sequence $(e^{(j)})_{j=1}^{\infty}$ cannot have a convergent subsequence. (Note that $C$ is closed, though.)

$D$ is not compact. The vectors $e^{(j)}$ defined in (1) all belong to $D$, so this counter-example works for $D$ as well.

(For the curious, note that in addition to not being totally bounded, the set $D$ is in fact not closed either. To show this, set $x^{(j)} = (1, 1/2, 1/3, 1/4, \ldots, 1/j, 0, 0, 0, \ldots)$, set $\beta_j = \sum_{n=1}^{j} \frac{1}{n}$, and set $y^{(j)} = \frac{1}{2\beta_j} x^{(j)}$. Then $y^{(j)} \in D$ for every $j$. But $y^{(j)} \to 0$ in $X$ (since $(x^{(j)})$ is a bounded sequence in $X$ and $\beta_j \to \infty$), and $0 \notin D$, so $D$ cannot be closed.)