Problem 4.1: Define a contour $\Gamma_1$ via
$$\Gamma_1 = \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + 2x_2^2 = 1 \}.$$ Let $\Omega_1$ denote the domain interior to $\Gamma_1$. Define points $a, b \in \Gamma_1$ and $c \in \Omega_1$ via
$$a = (1, 0), \quad b = (\cos(0.7), (1/\sqrt{2}) \sin(0.7)), \quad c = (0.3, 0.2).$$
Let $u$ be the unique solution to
\begin{align*}
-\Delta u(x) &= 0, \quad x \in \Omega_1, \\
u(x) &= f(x), \quad x \in \Gamma_1,
\end{align*}
where
$$f(x_1, x_2) = x_1 e^{\sin(10x_2)}.$$ Let $u$ have the representation
$$u(x) = [S \sigma](x) = \int_{\Gamma_1} \frac{1}{2\pi} \log \frac{R}{|x - x'|} \sigma(x') \, dl(x'),$$ where $R$ is chosen so that $R/|x - x'| \geq 2$ for all $x, x' \in \Gamma$. (Say $R = 10$.)

Your task is to form an equation for $\sigma$, discretize this equation, solve the equation, and then to evaluate the function $u$.

Let $N$ denote the number of degrees of freedom in your approximation, and include in your solution the following table (with values filled in where the question marks are):

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\sigma(a)$</th>
<th>$\sigma(b)$</th>
<th>$u(c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>200</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>400</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>800</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Include as large $N$ as your computer can handle in a reasonable amount of time, and estimate the convergence rate for each column.

**Hint:** In developing the code, it might be helpful to solve some Laplace problems for which you know the solution. For instance, set $v(x_1, x_2) = x_1^2 - x_2^2$ and $f = v(\Gamma)$. Or, pick a point $z$ outside of $\Gamma$, and set $v(x) = \log |x - z|$. (If you try the latter option, what happens if you pick $z$ very close to $\Gamma$?)

![The geometry of Problems 4.1 and 4.3.](image)
The geometry of Problems 4.2 and 4.4.

**Problem 4.2:** Repeat Problem 4.1, but now set\[ G_1(t) = 1.5 \cos(t) + 0.1 \cos(6t) + 0.1 \cos(4t), \]
\[ G_2(t) = \sin(t) + 0.1 \sin(6t) - 0.1 \sin(4t), \]
and define\[ \Gamma_2 = \{ x = (G_1(t), G_2(t)) : t \in [0, 2\pi) \}. \]
The Dirichlet data \( f \) is the same. Report \( \sigma(d) \) and \( u(e) \) for the points\[ d = (1.7, 0), \quad e = (0.3, 0.5). \]

**Problem 4.3:** Repeat Problem 4.1 (with the contour \( \Gamma_1 \)) but now use the Ansatz\[ u(x) = [D\sigma](x) = \int_{\Gamma_1} \frac{n(x') \cdot (x - x')}{2\pi|x - x'|^2} \sigma(x') \, dl(x') \]
where \( n(x') \) is the outwards pointing unit normal at \( x' \).

**Problem 4.4:** Repeat Problem 4.2 (with the contour \( \Gamma_2 \)) but now use the Ansatz\[ u(x) = [D\sigma](x) = \int_{\Gamma_2} \frac{n(x') \cdot (x - x')}{2\pi|x - x'|^2} \sigma(x') \, dl(x') \]
where \( n(x') \) is the outwards pointing unit normal at \( x' \).

**Hint:** In debugging your codes for the double layer potential, you may find the third Green identity useful (in particular that \( [D1](x) = -1 \) when \( x \in \Omega \), and \( [D1](x) = -1/2 \) when \( x \in \Gamma \)).

**Hint:** In my codes, I represent a contour \( \Gamma \) in an object \( C \) of size \( 6 \times N \), where \( N \) is the number of discretization points. Column \( i \) of \( C \) encodes the data for parameter point \( t_i \):
\[
\begin{align*}
C(1,i) &= G_1(t_i) \\
C(2,i) &= G_1'(t_i) \\
C(3,i) &= G_1''(t_i) \\
C(4,i) &= G_2(t_i) \\
C(5,i) &= G_2'(t_i) \\
C(6,i) &= G_2''(t_i)
\end{align*}
\]