Problem 1: Consider the function
\[ F : \mathbb{R}^2 \to \mathbb{R}^2 : (x_1, x_2) \mapsto (x_1 + 2 x_1 x_2, 1 + x_2 - 2 x_1^2) e^{-x_1^2 - x_2^2}. \]
Let \( \Omega \) denote the set of \( C^1 \) closed positively oriented contours in \( \mathbb{R}^2 \) that do not self-intersect. Determine
\[ \sup_{\gamma \in \Omega} \int_{\gamma} (F \cdot n) \, ds, \]
where \( n \) is the outwards pointing unit normal of \( \gamma \), and \( ds \) refers to parameterization using arc-length.

Problem 2: Determine
\[ \min_{a, b, c \in \mathbb{R}} \int_{-1}^{1} \left( x^3 - a - b x - c x^2 \right)^2 \, dx. \]
Hint: Recall that the first three Legendre polynomials are
\[ P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3 x^2 - 1). \]

Problem 3: Let \( S(\mathbb{R}) \) define the Schwartz functions on \( \mathbb{R} \), and let \( S^*(\mathbb{R}) \) denote the set of tempered distributions on \( \mathbb{R} \). Define for \( N = 1, 2, 3, \ldots \) the map \( T_N : S(\mathbb{R}) \to \mathbb{R} \) by
\[ \langle T_N, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{\cos(N x)}{x} \varphi(x) \, dx. \]
(a) Prove that \( T_N \in S^*(\mathbb{R}) \).
(b) Prove that the sequence \( (T_N)_{N=1}^\infty \) converges in \( S^*(\mathbb{R}) \).

Problem 4: Let \( A \) be an \( n \times n \) matrix, let \( b \) be a given vector in \( \mathbb{R}^n \), and consider the linear system of equations
\[ (1) \quad A x = b. \]
We write \( A = D + L + U \), where \( D \) is a diagonal matrix, \( L \) is a lower triangular matrix, and \( U \) is upper triangular. The so called Jacobi method for solving (1) starts with an initial guess \( x_0 \) and attempts to improve upon the initial guess via the iteration
\[ x_n = -D^{-1} \left( L + U \right) x_{n-1} + D^{-1} b. \]
Prove that if \( A \) is diagonally dominant, meaning that for any \( i = 1, 2, \ldots, n \) we have \(|a_{ii}| > \sum_{j \neq i} |a_{ij}|\), then the Jacobi iteration converges to the solution \( x \) of (1) for any initial guess \( x_0 \). State any theorems that you may use.

Problem 5: Let \( f \) be an integrable function on \( \mathbb{R} \) such that \( f \geq 0 \). Set
\[ g(t) = \int_{\mathbb{R}} \cos(t x) f(x) \, dx, \quad \text{for } t \in \mathbb{R}. \]
Show that \( g \) is twice differentiable if and only if \( \int_{\mathbb{R}} x^2 f(x) \, dx < \infty \).