Problem 1. A transmitter sends a message by using a binary code, namely, a sequence of 0's and 1's. Each transmitted bit (0 or 1) must pass through three relays to reach the receiver.

At each relay, the probability is 0.20 that the bit sent will be different from the bit received (a reversal). Assume that the relays operate independently of one another.

Transmitter → Relay 1 → Relay 2 → Relay 3 → Receiver

a. If a 1 is sent from the transmitter, what is the probability that a 1 is sent by all three relays?

b. If a 1 is sent from the transmitter, what is the probability that a 1 is received by the receiver?

c. Suppose that 70% of all bits sent from the transmitter are 1s (and only 30% are 0’s). If a 1 is received by the receiver, what is the probability that a 1 was in fact sent by the transmitter?

The probability of a bit reversal is 0.2, so the probability of maintaining a bit is 0.8.

a. Using independence, \( P(\text{all three relays correctly send 1}) = (0.8)(0.8)(0.8) = 0.512. \)

b. In the accompanying tree diagram, each 2 indicates a bit reversal (and each 8 its opposite). There are several paths that maintain the original bit: no reversals or exactly two reversals (e.g., 1 → 1 → 0 → 1, which has reversals at relays 2 and 3). The total probability of these options is 0.512 + (0.8)(0.2)(0.2) + (0.2)(0.8)(0.2) + (0.2)(0.2)(0.8) = 0.512 + 3(0.032) = 0.508.

The diagram shows:

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  1
 / \             2
/   \               8
1    0               2
       \           0.032
         \          0.032
           \        0.8
             \       0.2
               \     2
                 \   0.8
                   \  0.2
                     \ 0.512
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c. Using the answer from b, \( P(1 \text{ sent} | 1 \text{ received}) = \frac{P(1 \text{ sent} \cap 1 \text{ received})}{P(1 \text{ received})} = \frac{0.512}{0.512 + 0.032 + 0.032} = 0.4256 \). (Note: This calculation is incorrect as it should be \( \frac{0.512}{0.512 + 0.032 + 0.032} = 0.4256 \).)

The correct calculation is:

\[
P(1 \text{ sent} | 1 \text{ received}) = \frac{P(1 \text{ sent} \cap 1 \text{ received})}{P(1 \text{ received})} = \frac{0.512}{0.512 + 0.032 + 0.032} = 0.4256 \]

In the denominator, \( P(1 \text{ received} | 0 \text{ sent}) = 1 - P(0 \text{ received} | 0 \text{ sent}) = 1 - 0.608, \) since the answer from b also applies to a 0 being relayed as a 0.
Problem 2. Disregarding the leap year possibility (a birthday on Feb 29), suppose a randomly selected person is equally like to have been born on any of the other 365 days.

a) If 10 people are randomly selected, what is the probability that all have different birthdays? That at least two have the same birthday?

b) With $k$ replacing 10 in part (a), what is the smallest $t$ for which there is at least a 50% chance that two or more people will have the same birthday?

c) If ten people are randomly selected, what is the probability that either at least two have the same birthday or at least two have the same last three digits of their Social Security numbers?

This is the famous "Birthday Problem" in probability.

a. There are $365^{10}$ possible lists of birthdays, e.g. (Dec 10, Sep 27, Apr 1, ...). Among those, the number with zero matching birthdays is $P_{10,365}$ (sampling ten birthdays without replacement from 365 days). So,

$$P(\text{all different}) = \frac{P_{10,365}}{365^{10}} = \frac{(365)(364) \cdots (356)}{(365)^{10}} = .883.$$  

$P(\text{at least two the same}) = 1 - .883 = .117.$

b. The general formula is $P(\text{at least two the same}) = 1 - \frac{P_{k,365}}{365^k}$. By trial and error, this probability equals .476 for $k = 22$ and equals .507 for $k = 23$. Therefore, the smallest $k$ for which $k$ people have at least a 50-50 chance of a birthday match is 23.

c. There are 1000 possible 3-digit sequences to end a SS number (000 through 999). Using the idea from a, 

$$P(\text{at least two have the same SS ending}) = 1 - \frac{P_{10,999}}{1000^{10}} = 1 - .956 = .044.$$  

Assuming birthdays and SS endings are independent, 

$$P(\text{at least one "coincidence"}) = P(\text{birthday coincidence} \cup \text{SS coincidence}) = .117 + .044 - (.117)(.044) = .156.$$
Problem 3. Vehicles arrive at an Emissions testing station according to a Poisson distribution with the rate \( \lambda = 10 \) per hour. Suppose there is a probability of 0.5 that a vehicle will have some sort of violation.

a) What is the probability that exactly 10 vehicles arrive during an hour, and that all 10 have no emission violations?

b) For any fixed \( y > 10 \), what is the probability that \( y \) arrive during the hour, and out of those \( y \) vehicles, 10 vehicles have no violations?

c) What is the probability that 10 "no-violation" vehicles arrive during the next hour?

a. Let \( Y \) denote the number of cars that arrive in the hour, so \( Y \sim \text{Poisson}(10) \). Then

\[
P(Y = 10 \text{ and no violations}) = P(Y = 10) \cdot P(\text{no violations} | Y = 10) = \frac{e^{-10}10^{10}}{10!} \cdot (0.5)^{10},
\]

assuming the violation statuses of the 10 cars are mutually independent. This expression equals 0.00122.

b. Following the method from a, \( P(y \text{ arrive and exactly 10 have no violations}) = P(y \text{ arrive}) \cdot P(\text{exactly 10 "successes" in } y \text{ trials when } p = .5) = \frac{e^{-10}10^{y}}{y!} \cdot \left( \frac{1}{2} \right)^{y}. \]


c. \( P(\text{exactly 10 without a violation}) = \sum_{r=16}^{\infty} \frac{e^{-10}5^{r}}{10!(r-10)} = \frac{e^{-10} \cdot 5^{10}}{10!} \sum_{r=16}^{\infty} \frac{5^{r-10}}{(r-10)!} = \frac{e^{-10} \cdot 5^{10}}{10!} \cdot \sum_{r=0}^{\infty} \frac{5^{r}}{r!} = p_{10; 5} \).

In fact, generalizing this argument shows that the number of "no-violation" arrivals within the hour has a Poisson distribution with mean parameter equal to \( \mu = ap = 10 \cdot 0.5 = 5 \).
Problem 4. Let $X$ denote the lifetime of a component with $f(x)$ and $F(x)$ being the pdf and cdf of $X$ respectively. The probability that the component fails in the interval $(x, x+\Delta x)$ can be computed approximately as $f(x)\Delta x$. The conditional probability that a component fails in that interval $(x, x+\Delta x)$, given that the component has lasted at least until time $x$, is then $f(x) \Delta x / [1-F(x)]$.

Dividing this last expression by $\Delta x$ produces the new quantity, called the failure rate or the hazard function:

$$ r(x) = \frac{f(x)}{1-F(x)} $$

An increasing hazard function means that the older components are increasingly likely to wear out and fail, and a decreasing failure rate can be viewed as evidence of increasing reliability with age.

a) If $X$ has an Exponential distribution with parameter $\lambda$, derive $r(x)$.

b) If $X$ has a Weibull distribution with parameters $\alpha$ and $\beta$, what is $r(x)$? For what values of $\alpha$ and $\beta$ will $r(x)$ be increasing? For what values $\alpha$ and $\beta$ will $r(x)$ be decreasing with $x$?

c) Since $r(x)$ can also be found by differentiating $\ln[1-F(x)]$, as $-(d/dx) \ln[1-F(x)]$, it follows that $F(x)$ can also be obtained from $r(x)$ by integrating and exponentiating. If we are given the hazard function:

$$ r(x) = a(1-x/b) $$

for $x$ between 0 and $b$ (and assuming that the hazard function is 0 elsewhere), what is the cdf and pdf of $X$?

a. $f(x) = \lambda e^{-\lambda x}$ and $F(x) = 1 - e^{-\lambda x}$, so $r(x) = \frac{\lambda e^{-\lambda x}}{1 - (1 - e^{-\lambda x})} = \frac{\lambda e^{-\lambda x}}{e^{-\lambda x}} = \lambda$, a constant; this is consistent with the memoryless property of the exponential distribution.

b. For the Weibull distribution, $r(x) = \frac{f(x)}{1-F(x)} = \left(\frac{\alpha}{\beta^x}\right)x^{\alpha-1}$. For $\alpha > 1$, $r(x)$ is increasing (since the exponent on $x$ is positive), while for $\alpha < 1$ it is a decreasing function.

c. $\ln(1-F(x)) = -(r(x) dx - \frac{\alpha}{\beta} x^{\alpha-1})dx = -\alpha \left( x - \frac{x^2}{2\beta} \right) \Rightarrow F(x) = 1 - e^{-\frac{x^2}{2\beta}}$

$$ f(x) = F'(x) = \alpha \left(1 - \frac{x}{\beta}\right) e^{-\frac{x^2}{2\beta}} \text{ for } 0 \leq x \leq \beta. $$