1. Use Green’s theorem to show that the centroid of a region $D$ bounded by a simple closed path $C$ can be calculated using the following formulas:

$$\bar{x} = \frac{1}{2A} \int_C x^2 dy$$
$$\bar{y} = -\frac{1}{2A} \int_C y^2 dx$$

**Solution 1.** The centroid of a circle is defined:

$$\bar{x} = \frac{1}{A} \int \int_R x dA \quad \bar{y} = \frac{1}{A} \int \int_R y dA,$$
where $A = \int \int_R dA$.

To find the $x$-coordinate define $F = \langle P, Q \rangle = \langle 0, \frac{1}{2}x^2 \rangle$ so that $Q_x - P_y = x$. Applying Green’s Theorem:

$$\bar{x} = \frac{1}{A} \int \int_R x dA = \frac{1}{A} \left( \int_C Pdx + \int_C Qdy \right) = \frac{1}{A} \int_C \frac{1}{2}x^2 dy = \frac{1}{2A} \int_C x^2 dy$$

To find the $y$-coordinate define $F = \langle -\frac{1}{2}y^2, 0 \rangle$ so that $Q_x - P_y = y$ and follow the same process:

$$\bar{y} = \frac{1}{A} \int \int_R y dA = -\frac{1}{2A} \int_C y^2 dx$$

2. Among all rectangular regions such that $0 \leq x \leq a$ and $0 \leq y \leq b$, find the values of $a$ and $b$ for which the total outward flux of the vector field $F = \langle x^2 + 4xy, -6y \rangle$ across all four sides has the least value. What is the least flux?

**Solution 2.** Use the vector form of Green’s theorem:

$$\text{flux} = \int_C F \cdot n ds = \int \int_R \nabla \cdot F dA$$

The region is the rectangle defined by $0 \leq x \leq a, 0 \leq y \leq b$. $\nabla \cdot F = P_x + Q_y = 2x + 4y - 6$, so we can find the flux as a function $f(a,b)$:
\[ f(a, b) = \int_0^a \int_0^b (2x + 4y - 6) dy dx = \int_0^a (2bx + 2b^2 - 6b) dx = a^2b + 2ab^2 - 6ab \]

To find the minimum flux, we find the critical points:

\[ 0 = f_a = 2ab + 2b^2 - 6b = b(2a + 2b - 6) \Rightarrow 2a + 2b = 6 \text{ since } b \neq 0 \]
\[ 0 = f_b = a^2 + 4ab - 6a = a(a + 4b - 6) \Rightarrow a + 4b = 6 \text{ since } a \neq 0 \]

Solving this system, we get \( a=2 \) and \( b=1 \). To make sure that this is a minimum we do a second derivative test:

\[ f_{aa}|_{(2,1)} = 2b|_{(2,1)} = 2 \]
\[ f_{bb}|_{(2,1)} = 4a|_{(2,1)} = 8 \]
\[ f_{ab}|_{(2,1)} = 2a + 4b - 6|_{(2,1)} = 2 \]
\[ D|_{(2,1)} = (2)(8) - (2)^2 = 12 \]

Since \( D > 0 \) and \( f_{aa} > 0 \), the minimum flux occurs at \( a = 2 \) and \( b = 1 \).

3. Suppose you place two identical coins, A and B, of radius R side by side. While holding coin A stationary, you roll coin B around the outside of coin A such that the edges of the coins do not slip. As you do this, a point P on the edge of coin B traces out a curve called an epicycloid.

(a) Determine a parameterization of the path of point P.
(b) Calculate the area enclosed by the path of point P.

**Solution 3.** (a) The coins can be described by the picture:
So the path of $P$ can be described by the parametrization:

$$r(\theta) = \langle 2R \cos \theta - r \cos(\theta + \alpha), 2R \sin \theta - r \sin(\theta + \alpha) \rangle$$

Since the coins are the same we have that $R = r$ which means $\alpha = \theta$, so this reduces to:

$$r(\theta) = R(2\cos \theta - \cos 2\theta, 2\sin \theta - \sin 2\theta)$$

For more details on the construction of the parametrization of an epicycloid, see http://en.wikipedia.org/wiki/Epicycloid

(b) We can apply Green’s Theorem with $F(x, y) = \langle P, Q \rangle = \langle 0, x \rangle$ and rewrite the problem as a line integral:

$$\int \int_R dA = \int \int_R (Q_x - P_y) dA = \int_C P dx + \int_C Q dy = \int_C x dy$$

Rewrite this in terms of $r(\theta)$ and evaluate using the identities $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\cos^3 \theta = \frac{1}{4}(3 \cos \theta + \cos 3\theta)$:

$$\int_0^{2\pi} x(\theta)y'(\theta)d\theta =$$

$$R^2 \int_0^{2\pi} (2 \cos \theta - \cos 2\theta)(2 \cos \theta - 2 \cos 2\theta)d\theta =$$

$$R^2 \int_0^{2\pi} (4 \cos^2 \theta - 6 \cos \theta \cos 2\theta + 2 \cos^2 2\theta)d\theta =$$

$$R^2 \int_0^{2\pi} (3 + 6 \cos \theta - 12 \cos^3 \theta + 2 \cos 2\theta + \cos 4\theta)d\theta =$$

$$R^2 \int_0^{2\pi} (3 - 3 \cos \theta + 2 \cos 2\theta - 3 \cos 3\theta + \cos 4\theta)d\theta =$$

$$R^2 \left[ 3\theta - 3 \sin \theta + \sin 2\theta - \sin 3\theta + \frac{1}{4} \sin 4\theta \right]_0^{2\pi} =$$

$$6\pi R^2$$