INSTRUCTIONS: Books, notes, crib sheets, and electronic devices are not permitted. Write your (1) name, (2) instructor’s name, and (3) recitation number on the front of your bluebook. Work all problems. Show and explain your work clearly. Note that a correct answer with incorrect or no supporting work may receive no credit, while an incorrect answer with relevant work may receive partial credit.

1. (25 points) Consider a spherical loaf of bread with radius \( R \). Suppose you cut a slice from the loaf between the planes \( y = a \) and \( y = b \), where \( 0 < a < b < R \). Set up, but do not evaluate, the integral that will give you the volume of the slice of bread.

2. (25 points) You have this strange desire to find the value of the integral \( I = \iint_R \left( \frac{x-y}{x+y} \right) \, dx \, dy \) over the region \( R \) in the first quadrant bounded by \( x + y = 2 \), \( x = 0 \) and \( y = 0 \). You decide to try the substitution \( u = x - y \) and \( v = x + y \).

   (a) Plot the region of integration in the \( x-y \) plane.

   (b) Solve for \( x \) and \( y \) in terms of \( u \) and \( v \) using the given substitution. Being the clever student that you are, you know that the rest of the problem depends on this calculation, so you make a point of checking your result. Right?

   (c) Plot the region of integration in the \( u-v \) plane.

   (d) Set up the integral in terms of \( u \) and \( v \).

   (e) Evaluate the integral in terms of \( u \) and \( v \) to determine the value of \( I \). Hint: think integration by parts!

3. (25 points) Suppose the mass of an object is given by

\[
M = \int_{\theta=0}^{2\pi} \int_{r=1}^{2} \int_{z=0}^{r} (z^2 + r^2) \, r \, dz \, dr \, d\theta.
\]

   (a) Determine the density \( \delta \) (mass per unit volume) of the object in terms of the cartesian coordinates \( x \), \( y \), and \( z \).

   (b) Plot the cross-section of the object in the \( r-z \) plane (this is a constant \( \theta \) plane in cylindrical coordinates) clearly labeling the boundaries of the object.

   (c) Rewrite the integral for \( M \) in the order \( dr \, dz \, d\theta \).

   (d) Convert the integral for \( M \) to spherical coordinates using the order \( d\phi \, d\rho \, d\theta \).

   (e) Evaluate one of your integrals to determine the mass of \( M \). (Yes, you may evaluate the original integral.)

4. (25 points) Consider a vector field in the \( x-y \) plane, \( \mathbf{F}(x,y) = xy \, \mathbf{i} + xy \, \mathbf{j} \), and a path \( P_1 \) that consists of two straight line segments. The first segment goes from point \( A(0,0) \) to point \( B(1,0) \), and the second segment goes from point \( B \) to point \( C(1,1) \).

   (a) Calculate the flow of \( \mathbf{F} \) along the entire of path \( P_1 \) beginning at \( A \) and ending at \( C \).

   (b) To get from \( A \) to \( C \), the particle could have taken a shorter path, \( P_2 \), by traversing part of the curve parameterized by \( \mathbf{r}(\tau) = \cos \tau \, \mathbf{i} + (1 + \sin \tau) \, \mathbf{j} \). A sketch of the path may help, but is not required. What is the flux of \( \mathbf{F} \) across \( P_2 \)?

   (c) Not including the letters used on this exam, write your favorite greek letter. Be sure to box your answer. Then, clearly explain why the value that you calculated in part (b) does not depend on how \( P_2 \) is parameterized.
Projections and distances \( \text{proj}_A \mathbf{B} = \left( \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A} \quad d = \frac{|\mathbf{P} \mathbf{S} \times \mathbf{v}|}{|\mathbf{v}|} \quad d = |\mathbf{P} \mathbf{S} \cdot \mathbf{n}| \)

Arc length, frenet formulas, and tangential and normal acceleration components

\[
\begin{align*}
\frac{ds}{dt} &= |\mathbf{v}| & \mathbf{T} &= \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} & \mathbf{N} &= \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/ds|} & \mathbf{B} &= \mathbf{T} \times \mathbf{N} \\
\frac{d\mathbf{T}}{ds} &= \kappa \mathbf{N} & \frac{d\mathbf{B}}{ds} &= -\tau \mathbf{N} & \kappa &= |\frac{d\mathbf{T}}{ds}| = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{1 + (f'(x))^2}^{3/2} = \frac{|\dot{x} \ddot{y} - \ddot{x} \dot{y}|}{\dot{x}^2 + \ddot{y}^2}^{3/2} & \tau &= -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} \\
\mathbf{a} &= a_N \mathbf{N} + a_T \mathbf{T} & a_T &= \frac{d|\mathbf{v}|}{dt} & a_N &= \kappa |\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}
\end{align*}
\]

Directional derivative, discriminant, and Lagrange multipliers

\[
\frac{df}{ds} = (\nabla f) \cdot \mathbf{u} \quad f_{xx}f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0
\]

Polar coordinates \( x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta \)

Cylindrical and spherical coordinates

\[
\begin{array}{|c|c|c|}
\hline
\text{Cylindrical to Rectangular} & \text{Spherical to Cylindrical} & \text{Spherical to Rectangular} \\
\hline
x = r \cos \theta & r = \rho \sin \phi & x = \rho \sin \phi \cos \theta \\
y = r \sin \theta & z = \rho \cos \phi & y = \rho \sin \phi \sin \theta \\
z = z & \theta = \theta & z = \rho \cos \phi \\
\hline
\end{array}
\]

\[
dV = dx \, dy \, dz = r \, dr \, d\theta \, dz = \rho^2 \sin \phi \, dp \, d\phi \, d\theta
\]

Substitutions in multiple integrals

\[
\iint_R f(x, y) \, dx \, dy = \iiint_G f(x(u, v), y(u, v)) \, |J(u, v)| \, du \, dv \quad \text{where} \quad J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v}
\]

Mass, moments, and center of mass

\[
\text{Mass} \quad M = \int \int_R \delta \, dA \\
\text{Moments} \quad M_x = \int \int_R y \, \delta \, dA \quad M_y = \int \int_R x \, \delta \, dA \quad \text{Center of mass} \quad \bar{x} = M_y / M \quad \bar{y} = M_x / M
\]

Flow and flux

\[
\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_C \mathbf{F} \cdot \mathbf{V} \, dt = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M \, dx + N \, dy \\
\text{Flux} = \int_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_C M \, dy - N \, dx
\]