1. The integrals:

\[ V = \int_0^1 \int_0^z \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} dydxdz + \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^1 \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} dzdydx \]

calculate the volume of an object

(a) Express \( V \) as one triple integral in cartesian coordinates using the order \( dzdxdy \).

(b) Express \( V \) in cylindrical coordinates using the order \( dzdrd\theta \).

(c) Express \( V \) in spherical coordinates using the order \( d\rho d\phi d\theta \).

(d) Evaluate one of your integrals.

**Solution 1.** (a) The first triple integral \( \int_0^1 \int_0^z \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} dydxdz \) represents a half-cone centered at the origin and extending upward along the z-axis until meeting the plane \( z = 1 \). This half cone is bounded by the y-axis plane, or in other words, the \( x = 0 \) plane.

Similarly, the second integral \( \int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^1 \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} dzdydx \) represents the top part of a sphere of radius \( \sqrt{2} \) that is centered at the origin. This second triple integral only represents the portion of the sphere above the \( z = 1 \) plane, as indicated by the innermost integral. Also, similar to the cone, only the half of the top of the sphere that is on the positive side of the x-axis is being figured in. The first integral in the second triple integral set tells us that. So to recap, the region in question looks somewhat like half of an ice cream cone that has been cut length-wise along the \( x = 0 \) plane.
\[ V = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz
dx\,dy \]

(b) In cylindrical coordinates: \(dz\,rd\theta\), our \(\theta\) bounds would be along
the y-axis, so \(\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\). Also \(0 \leq r \leq 1\) and we keep the same \(z\)
bounds from part a, however we use the formula \(r^2 = x^2 + y^2\) to
to change from \(x\) and \(y\) to \(r\). So \(r \leq z \leq \sqrt{2-r^2}\).

\[ V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} \int_{r}^{\sqrt{2-r^2}} rdz\,dr\,d\theta \]

(c) In spherical coordinates: \(d\rho d\phi d\theta\), the \(\theta\) bounds stay the same as
in part (b). Since the sides of the object are defined by a regular
cone, \(\phi\) will be bounded by \(\frac{\pi}{4}\). So \(0 \leq \phi \leq \frac{\pi}{4}\). \(\rho\) extends from the
origin to the top of the half sphere which is a maximum length of \(\sqrt{2}\).

\[ V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{4}} \int_{0}^{\sqrt{2}} \rho^2 \sin(\phi) d\rho d\phi d\theta \]

(d) Now, evaluating in spherical coordinates:

\[
V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{4}} \frac{1}{3} \rho^3 \sin(\phi) \theta^2 d\phi d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{4}} \frac{2\sqrt{2}}{3} \sin(\phi) d\phi d\theta \\
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{2\sqrt{2}}{3} \cos(\phi) \bigg|_{0}^{\frac{\pi}{4}} d\theta = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2\sqrt{2}}{3} \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{3} (1) d\theta \\
= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{3} - \frac{2\sqrt{2}}{3} d\theta = (-\frac{2}{3} + \frac{2\sqrt{2}}{3}) \theta \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
= (\frac{2\sqrt{2}}{3} - \frac{2}{3}) \pi
\]
2. Find the area enclosed by the ellipse $E$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the following way:

(a) First, write the double integral that gives the area of $E$ in cartesian coordinates. Don’t evaluate this integral!

(b) Next, transform the variables using $x = au$ and $y = bv$. Clearly sketch the domain of $E$ in the $uv$-plane. Rewrite the $xy$-integral from part (a) as an integral in the $uv$-plane.

(c) Evaluate the new integral from part (b) (feel free to change coordinate systems) and show that the area of $E$ is equal to $\pi ab$.

(d) Extend the above techniques to find the volume enclosed by the ellipsoid $E$ given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

**Solution 2.**

(a) \[ \int_{-a}^{a} \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} dy\,dx \]

(b) $x = au, y = bv$

$$\left(\frac{au}{a}\right)^2 + \left(\frac{bv}{b}\right)^2 = 1 \implies u^2 + v^2 = 1$$

The region maps onto the unit disk in the $uv$-plane.

$$J = \left| \begin{array}{cc} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{array} \right| = \left| \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right| = ab$$

So the integral becomes

$$ab \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} dv\,du$$
(c) Now we switch to polar coordinates: \( u = r \cos(\theta) \), \( v = r \sin(\theta) \)

\[
V = ab \int_0^{2\pi} \int_0^1 rdrd\theta = \frac{ab}{2} \int_0^{2\pi} d\theta = \pi ab
\]

(d) \( x = au, y = bv, z = cw \) and \( u^2 + v^2 + w^2 = 1 \) which is the unit sphere.

\[
J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc
\]

Now we switch to spherical coordinates:

\[
V = abc \int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta = \frac{abc}{3} \int_0^{2\pi} \int_0^\pi \sin \phi d\phi d\theta
\]

\[
= \frac{abc}{3} \int_0^{2\pi} [-\cos \phi]_0^\pi d\theta = \frac{2abc}{3} \int_0^{2\pi} d\theta = \frac{4\pi}{3} abc
\]