1. (25 points) A metal plate has mass density (mass per unit area) given by \( \delta(x,y) = \frac{x+y}{x^2 + y^2} \). The shape of the plate matches the region \( R \) in the first quadrant bounded by the \( y \)-axis, the curve \( y = x \) and the curve \( y = 2 - x \).

(a) Set up the integral(s) needed to calculate the area of \( R \) in Cartesian coordinates in the order \( x-y \).

(b) Set up the integral(s) needed to calculate the area of \( R \) in Cartesian coordinates in the order \( y-x \).

(c) Based on your results from either part (a) or (b), determine the area of \( R \).

(d) Set up the integral(s) needed to find the mass of the plate using polar coordinates in the order \( r-\theta \).

(e) Based on your result from part (d), determine the mass of the plate.

2. (25 points) Consider the integral 
\[
I = \int \int_R \left( 1 + \frac{y}{x} \right)^2 \, dy \, dx,
\]
where the region \( R \) is in the first quadrant of the \( x-y \) plane bounded by the curves \( x + y = 1 \), \( x + y = 4 \), \( y = x/2 \), and \( y = 2x \).

(a) Sketch the region \( R \) in \( x-y \) plane. Be sure to label all axes, boundaries, etc. on your sketch.

(b) Now consider the coordinate transform \( u = x + y \), \( v = y/x \). Solve for \( x \) and \( y \) in terms of \( u \) and \( v \) using the given substitution. Be sure to check this because the rest of the problem depends on this result!

(c) Transform the region \( R \) into its corresponding region \( S \) in the \( u-v \) plane. Make a clear sketch of the new region of integration \( S \) in the \( u-v \) plane. Be sure to label all axes, boundaries, etc. on your sketch.

(d) Calculate the Jacobian \( J(u,v) = \frac{\partial (x,y)}{\partial (u,v)} \) for the coordinate transform.

(e) Rewrite the integral for \( I \) over the region \( S \) in the \( u-v \) plane in terms of \( u \) and \( v \).

(f) Based on your result from part (e), determine the value of \( I \).

3. (25 points) For unknown reasons the Student Space Army (SSA) is continuing their unrelenting assault on Curtis Prime, and they have decided to build an orbiting space station from which to coordinate their attacks. The volume of the station is given by 
\[
V = 2\pi \int_{\theta=0}^{2\pi} \int_{z=0}^{\sqrt{4R^2-z^2}} \int_{r=R}^{\sqrt{4R^2-z^2}} r \, dr \, dz \, d\theta.
\]

(a) Plot the cross-section of the station in the \( r-z \) plane (this is a constant \( \theta \) plane in cylindrical coordinates) clearly labeling the boundaries of the object.

(b) Rewrite the integral in the order \( dz \, dr \, d\theta \).

(c) Convert the integral to spherical coordinates using the order \( d\phi \, d\rho \, d\theta \). Hint: \( \tan(\pi/6) = 1/\sqrt{3} \).

(d) Convert the integral to spherical coordinates using the order \( d\rho \, d\phi \, d\theta \).

(e) Evaluate one of your integrals to determine the value of \( V \).

4. (25 points) Consider the path in the \( x-y \) plane from the point \( A(0,0) \) to the point \( B(1,1) \) along the curve \( r(t) = t \, \textbf{i} + t^2 \, \textbf{j} \), and the vector function \( \textbf{F} = x^2 y \, \textbf{i} + y \, \textbf{j} \).

(a) Sketch the path in the \( x-y \) plane and set up, but do not evaluate, the integral to determine the length of the path from \( A \) to \( B \).

(b) Calculate the flow along the path from \( A \) to \( B \).

(c) Calculate the flux along the path from \( A \) to \( B \).

OVER
Projections and distances \[ \text{proj}_A B = \left( \frac{A \cdot B}{A \cdot A} \right) A \quad d = \frac{|PS \times v|}{|v|} \quad d = \frac{|PS \cdot n|}{|n|} \]

Arc length, frenet formulas, and tangential and normal acceleration components

\[ ds = |v| \, dt \quad T = \frac{dr}{ds} = \frac{v}{|v|} \quad N = \frac{dT/ds}{dT/dt} = \frac{dT/ds}{dT/dt} \quad B = T \times N \]

\[ \frac{dT}{ds} = \kappa N \quad \frac{dB}{ds} = -\tau N \quad \kappa = \frac{|v \times a|}{|v|^3} = \frac{|f''(x)|}{1 + (f'(x))^2}^{3/2} = \frac{|\dot{x}y - \dot{y}x|}{|\dot{x}|^3 + |\dot{y}|^3}^{3/2} \quad \tau = -\frac{dB}{ds} \cdot N \]

\[ a = a_N N + a_T T \quad a_T = \frac{d|v|}{dt} \quad a_N = \kappa |v|^2 = \sqrt{|a|^2 - a_T^2} \]

Directional derivative, discriminant, and Lagrange multipliers

\[ \frac{df}{ds} = (\nabla f) \cdot u \quad f_{xx} f_{yy} - (f_{xy})^2 \quad \nabla f = \lambda \nabla g, \quad g = 0 \]

Polar coordinates \[ x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2 \quad dA = dx \, dy = r \, dr \, d\theta \]

Cylindrical and spherical coordinates

<table>
<thead>
<tr>
<th>Cylindrical to Rectangular</th>
<th>Spherical to Cylindrical</th>
<th>Spherical to Rectangular</th>
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<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( r = \rho \sin \phi )</td>
<td>( x = \rho \sin \phi \cos \theta )</td>
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<tr>
<td>( y = r \sin \theta )</td>
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<td>( z = z )</td>
<td>( \theta = \theta )</td>
<td>( z = \rho \cos \phi )</td>
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\[ dV = dx \, dy \, dz = r \, dr \, d\theta \, dz = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

Substitutions in multiple integrals

\[ \int \int_R f(x, y) \, dx \, dy = \int \int_G f(x(u, v), y(u, v)) \, |J(u, v)| \, du \, dv \quad \text{where} \quad J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial u} - \frac{\partial y}{\partial u} \frac{\partial x}{\partial v} \]

Mass, moments, and center of mass \[ \text{Mass} \quad M = \int \int_R \delta \, dA \]

\[ \text{Moments} \quad M_x = \int \int_R y \, \delta \, dA \quad M_y = \int \int_R x \, \delta \, dA \quad \text{Center of mass} \quad \bar{x} = M_y / M \quad \bar{y} = M_z / M \]

Flow and flux \[ \text{Flow} = \int_C \mathbf{F} \cdot T \, ds = \int_C \mathbf{F} \cdot V \, dt = \int_C \mathbf{F} \cdot dr = \int_C M \, dx + N \, dy \]

\[ \text{Flux} = \int_C \mathbf{F} \cdot n \, ds = \int_C M \, dy - N \, dx \]