The population of a country is growing at a rate that is proportional to the population of the country. The population in 1990 was 20 million and in 2000 the population was 22 million. Estimate the population in 2020.

Solution: Let \( P(t) \) represent the population of the country at time \( t \), with \( t \) measured in years. We are given that the growth rate is proportional to the population; that is,

\[
P'(t) = rP(t)
\]

where \( r \) is the constant of proportionality. The solution to this differential equation is

\[
P(t) = P_0 e^{rt}
\]

where \( P_0 \) is the population at time 0. It’s easiest if we let \( t = 0 \) correspond to the year 1990. Then \( P_0 = 20 \) million.

This gives us just one unknown \( r \). We use the fact that the population in 2000 was 22 million. This corresponds to \( t = 10 \). So

\[
22,000,000 = 20,000,000 \cdot e^{10r}
\]

\[
\frac{11}{10} = e^{10r}
\]

\[
\ln\left(\frac{11}{10}\right) = 10r
\]

\[
\frac{\ln\left(\frac{11}{10}\right)}{10} = r
\]

This gives us the equation

\[
P(t) = 20,000,000 \cdot e^{\left(\frac{\ln\left(\frac{11}{10}\right)}{10}\right) t}
\]

This could be simplified... or we can just plug in \( t = 30 \) to find the estimate population in 2020.

\[
P(30) = 20,000,000 \cdot e^{\left(\frac{\ln\left(\frac{11}{10}\right)}{10}\right) 30}
\]

\[
= 26,620,000
\]

\[
\approx 27 \text{ million}
\]

2. We start with a tank containing 50 gallons of salt water with the salt concentration being 2 lb/gal. Salt water with a salt concentration of 3 lb/gal is then poured into the top of the tank at the rate of 3 gal/min
and salt water is at the same time drained from the bottom of the tank at the rate of 3 gal/min. We will consider the water and salt mixture in the tank to be well-stirred and at all times to have a uniform concentration of salt. Find the function $S(t)$ that gives the amount of salt in the tank as a function of time $(t)$ since we began pouring in salt water at the top and simultaneously draining salt water from the bottom of the tank. How long before there will be 120 pounds of salt in the tank?

**Solution:** Let $S(t)$ be the pounds of salt in the tank after $t$ minutes. We are given $S(0) = 100$. We know that

$$\frac{dS}{dt} = \text{rate of salt in} - \text{rate of salt out} = 9 - \frac{3}{50}S$$

So we have the linear nonhomogeneous differential equation

$$\frac{dS}{dt} + \frac{3}{50}S = 9$$

You can solve this DE using variation of parameters or an integrating factor. The solution is

$$S = 150 + Ce^{-\frac{3}{50}t}$$

Use the initial condition to solve for $C$, obtaining $C = -50$. Thus we have

$$S = 150 - 50e^{-\frac{3}{50}t}$$

Now we'd like to know at what time we have $S = 150$. So we solve for $t$ in

$$120 = 150 - 50e^{-\frac{3}{50}t},$$

obtaining

$$t = \frac{-50 \ln \frac{2}{9}}{3} \approx 8.51376.$$
Notice we can take the absolute value off of \( \sec(t) \) because of the condition \( 0 \leq t < \frac{\pi}{2} \).

Now multiply both sides of the DE by the integrating factor.

\[
\sec(t)y' + \sec(t) \tan(t)y = 2 \sec(t) \cos^2(t) \sin(t) - \sec^2(t),
\]

which simplifies to

\[
\frac{d}{dt} (\sec(t)y) = 2 \cos(t) \sin(t) - \sec^2(t)
\]

Integrate both sides to obtain

\[
\sec(t)y = -\frac{1}{2} \cos(2t) - \tan(t) + C
\]

(We used the identity \( 2 \cos(t) \sin(t) = \sin(2t) \).) Solve for \( y \).

\[
y = -\frac{1}{2} \cos(t) \cos(2t) - \sin(t) + C \cos(t)
\]

Now apply the initial condition to solve for \( C \). You should get \( C = 7 \).

So the solution to the initial value problem is

\[
y = -\frac{1}{2} \cos(t) \cos(2t) - \sin(t) + 7 \cos(t)
\]

4. Ignoring resistance, a sailboat starting from rest accelerates \( \left( \frac{dv}{dt} \right) \) at a rate proportional to the difference between the velocities of the wind and the boat.

(a) Write the velocity as a function of time if the wind is blowing at 20 ft/sec and after one second the boat is moving at 5 ft/sec. Assume the boat started from rest. (Hint: This problem is similar to Newton’s Law of Cooling.)

(b) Use the result in part (a) to write the distance traveled by the boat as a function of time. How far does the boat travel in the first 8 seconds?

Solution:

(a) Let \( v(t) \) represent the velocity of the boat, in feet per second, after \( t \) seconds. We are told that \( \frac{dv}{dt} \) is proportional to the difference between \( v \) and wind speed. Thus we can write

\[
\frac{dv}{dt} = k(v - 20),
\]
where \( k \) is some constant of proportionality. This DE is separable.

\[
\frac{1}{v - 20} \, dv = k \, dt
\]

\[
\ln |v - 20| = kt + C
\]

\[
|v - 20| = C_2 e^{kt}
\]

\[
v(t) = 20 + C_2 e^{kt}
\]

Now we know that \( v(0) = 0 \). This allows us to solve for \( C_2 \). We obtain \( 0 = 20 + C_2 \), so \( C_2 = -20 \). Then we can solve for \( k \), using \( v(1) = 5 \). We have \( 5 = 20 - 20e^k \), so \( k = \ln\left(\frac{3}{4}\right) \). Now we can write the formula for \( v(t) \) as

\[
v(t) = 20 - 20 \left( e^{\ln\left(\frac{3}{4}\right)} \right)^t
\]

\[
= 20 - 20 \left( \frac{3}{4} \right)^t
\]

(b) Let \( s(t) \) represent the distance in feet traveled by the boat after \( t \) seconds. It is clear that \( s(0) \) must be 0. Also, recall from calculus that \( s(t) = \int v(t) \, dt \). So

\[
s(t) = \int \left( 20 - 20 \left( \frac{3}{4} \right)^t \right) \, dt
\]

\[
= 20t - \frac{20}{\ln\left(\frac{3}{4}\right)} \left( \frac{3}{4} \right)^t + C
\]

Solve for \( C \) using \( s(0) = 0 \); obtain \( C = \frac{20}{\ln\left(\frac{3}{4}\right)} \). Then

\[
s(t) = 20t - \frac{20}{\ln\left(\frac{3}{4}\right)} \left( \frac{3}{4} \right)^t + \frac{20}{\ln\left(\frac{3}{4}\right)}.
\]

The distance traveled in the first 8 seconds is \( s(8) \approx 97.44 \) feet.

5. The following three figures show direction fields for first-order differential equations. One of these corresponds to a nonlinear DE, one is linear homogeneous, and one is linear nonhomogeneous. Which is which? Explain how you can tell.

Hints: It is a fact that \( y = 0 \) is always a solution to a linear homogeneous DE. (Why is this?)

To help you identify linear DEs: Remember that for any linear DE, the sum of two solutions is also a solution; and any constant multiple of a solution is also a solution. Also, for a fixed \( t \) value, \( y' \) must be a linear function of \( y \) (since the DEs are first-order.)
Solution: (a) is nonlinear, (b) is linear nonhomogeneous, and (c) is linear homogeneous.