1. The nonlinear system
\[
\begin{align*}
    x^2 - y - 1 &= 0 \\
    (x - 2)^2 + (y - \frac{1}{2})^2 - 1 &= 0
\end{align*}
\]
has two real solutions.

   a. Implement Newton’s method for systems, and apply it to this system of equations, starting iterations with (0,0) and with (2,2). Print out the successive iterates to quite high precision (format long), so you can see how the convergence proceeds.

   b. Choose some complex starting point, and see if you can catch one of its two complex solutions (you’ll then find the second solution by conjugating both \(x\) and \(y\)).

2. Consider the system
\[
\begin{align*}
    x + 1 + (x+y)^2 - 4x &= 0 \\
    y + 1 + (x-y)^2 - 4y &= 0
\end{align*}
\]
Find a region \(D\) in the \(xy\)-plane for which the fixed point iteration
\[
\begin{align*}
    x_{n+1} &= \frac{1}{\sqrt{2}} \sqrt{1+(x_n+y_n)^2 - \frac{2}{3}} \\
    y_{n+1} &= \frac{1}{\sqrt{2}} \sqrt{1+(x_n-y_n)^2 - \frac{2}{3}}
\end{align*}
\]
is guaranteed to converge to a unique solution for any start point \((x_0, y_0)\) in \(D\). State clearly what properties this region must have.

3. Let \(f(x, y)\) be a smooth function such that \(f(x, y) = 0\) defines a smooth curve in the \(xy\)-plane. We want to find some point on this curve that lies in the neighborhood of a start guess \((x_0, y_0)\) that is off the curve.

   a. Derive the iteration scheme
\[
\begin{align*}
    x_{n+1} &= x_n - d f_x \\
    y_{n+1} &= y_n - d f_y
\end{align*}
\]
for solving the task outlined above. Here \(d = f/(f_x^2 + f_y^2)\), and it is understood that \(f, f_x, f_y\) are all evaluated at the location \((x_n, y_n)\).

   **Hint:** One way to proceed is to look for a new iterate \((x_{n+1}, y_{n+1})\) that (i) lies on the gradient line through \((x_n, y_n)\) and (ii) also obeys \(f(x, y) = 0\). Apply Newton to this \(2\times2\) system.

   b. The iteration scheme above generalizes in an obvious way to moving from a start location \((x_0, y_0, z_0)\) onto a surface \(f(x, y, z) = 0\). With use of this iteration, find a point on the ellipsoid \(x^2 + 4y^2 + 4z^2 = 16\) when starting from \(x_0 = y_0 = z_0 = 1\). Give numerical evidence showing that the iteration indeed is quadratically convergent.

4. Apply the Sturm sequence technique to check how many real zeros the polynomial \(x^4 - x^3 - 2x^2 - 2x + 4\) has in the interval \([-2, 2]\), and also on the full real axis.
The following is an example that illustrates how one uses the Sturm's sequence method to locate polynomial roots along the real line:

**Example:** Let \( f(x) = x^6 + 4x^5 + 4x^4 - x^2 - 4x - 4 \) \((= (x^2+1)(x^2-1)(x+2)^2)\).

Consider then the following sequence of polynomials of successively decreasing orders:

\[
\begin{align*}
  f_1 &= f(x) = x^6 + 4x^5 + 4x^4 - x^2 - 4x - 4 \\
  f_2 &= f'' = 6x^5 + 20x^4 + 16x^3 - 2x - 4 \\
  f_3 &= \left\{ \begin{array}{ll}
  & \text{negative of remainder in long division } f_1 / f_2 \\
  & \text{(scale to get 'nice' integer coefficients if so desired)} \\
\end{array} \right.
\quad = 4x^4 + 8x^3 + 3x^2 + 14x + 16 \\
  f_4 &= \left\{ \begin{array}{ll}
  & \text{same as above, but based on } f_2 / f_3 \\
\end{array} \right.
\quad = x^3 + 6x^2 + 12x + 8 \\
  f_5 &= \left\{ \begin{array}{ll}
  & \text{... based on } f_3 / f_4 \\
\end{array} \right.
\quad = -17x^2 - 58x - 48 \\
  f_6 &= \left\{ \begin{array}{ll}
  & \text{... based on } f_4 / f_5 \\
\end{array} \right.
\quad = -x - 2 \\
  f_7 &= \left\{ \begin{array}{ll}
  & \text{... based on } f_5 / f_6 \\
\end{array} \right.
\quad = 0
\end{align*}
\]

Next, evaluate the signs of these polynomials at some set of points, for ex. at \( x = -\infty, +\infty, 0, -24/17 \). We get

<table>
<thead>
<tr>
<th></th>
<th>(-\infty)</th>
<th>(+\infty)</th>
<th>0</th>
<th>-24/17</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 )</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( f_2 )</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( f_3 )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( f_4 )</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( f_5 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>( f_6 )</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( f_7 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

where the bottom line shows the number of sign changes there is in each column. The difference between any of these numbers tells how many distinct zeros (i.e. ignoring multiplicity) \( f(x) \) has in the corresponding interval. In the present example, there are thus 3 distinct real roots altogether (between \(+\infty\) and \(-\infty\) ), one of which is seen to be located between 0 and \(+\infty\) , etc.