1. (20 Points) Veronica the vampire, much to her distress, still has a single window that lets in daylight. It is in the shape of the portion of the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \) that is in the first quadrant. Deep in her dark basement Veronica has many perfectly cut rectangles of black glass of all sizes. What are the dimensions of the rectangle that fits inside of the window so it blocks the most light from coming into her house?

Solution: Consider \( f(x, y) = xy \) subject to the constraint \( g(x, y) = \frac{x^2}{4} + \frac{y^2}{9} = 1 \) (along with \( x > 0, y > 0 \)). The partial derivatives of \( f \) and \( g \) are

\[
\begin{align*}
    f_x &= y, \\
    f_y &= x, \\
    g_x &= \frac{x}{2}, \\
    g_y &= \frac{2}{9}y
\end{align*}
\]

Using the method of Lagrange multipliers it follows that

\[
\begin{align*}
    y &= \frac{\lambda}{2}x, \\
    x &= \frac{2\lambda}{9}y, \\
    1 &= \frac{x^2}{4} + \frac{y^2}{9}
\end{align*}
\]

Substituting the first equation into the second equation implies that either \( x = 0 \) or \( \lambda = 3 \). Only consider the second case since \( x \) needs to be positive. If \( \lambda = 3 \) then

\[
\begin{align*}
    \frac{x^2}{4} + \frac{9}{4} \frac{x^2}{9} &= 1 \\
    \Rightarrow x^2 &= 2
\end{align*}
\]

Thus,

\[
[x = \sqrt{2}, \quad y = \frac{3}{2} \sqrt{2}]
\]

(Or the dimensions are \( \sqrt{2} \times 3\sqrt{2}/2 \).)

2. (20 Points) Consider the function defined by \( f(x, y) = \ln(1 + 2x - y) \).

(a) Find the second-order Taylor polynomial \( P_2(x, y) \) approximation of \( f(x, y) \) centered at \( (x, y) = (0, 0) \).

Solution: A second degree Taylor polynomial is given by

\[
f(x, y) = f(x_0, y_0) + \left[(x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0)\right] + \frac{1}{2!} \left[(x - x_0)^2f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2f_{yy}(x_0, y_0)\right]
\]

For this problem

\[
\begin{align*}
    f_x &= \frac{2}{1 + 2x - y} \\
    f_y &= \frac{-1}{1 + 2x - y} \\
    f_{xx} &= \frac{-4}{(1 + 2x - y)^2} \\
    f_{xy} &= \frac{2}{(1 + 2x - y)^2} \\
    f_{yy} &= \frac{-1}{(1 + 2x - y)^2}
\end{align*}
\]
Thus,

\[ P_2(x, y) = 2x - y + \frac{1}{2}(-4x^2 + 4xy - y^2) \]

(b) If \(|x| \leq 0.1\) and \(|y| \leq 0.2\), find an upper bound on the error associated with using \(P_2(x, y)\) to approximate \(f(x, y)\).

**Solution:** The error is given by

\[ |E(x, y)| \leq \frac{M}{3!}(|x - x_0| + |y - y_0|)^3 \]

where \(\{|f_{xxx}|, |f_{xyy}|, |f_{xxy}|, |f_{yyy}|\} \leq M\). It follows that

\[
\begin{align*}
  f_{xxx} & = \frac{16}{(1 + 2x - y)^3} \\
  f_{xyy} & = \frac{4}{(1 + 2x - y)^3} \\
  f_{xxy} & = -\frac{8}{(1 + 2x - y)^3} \\
  f_{yyy} & = -\frac{2}{(1 + 2x - y)^3}
\end{align*}
\]

Observe that the partials are all maximized at the point \((-0.1, 0.2)\) because this minimizes the denominator. Therefore \(M = 16/(0.6)^3\), which gives an error of

\[ E \leq \frac{16}{6(0.6)^3}(0.1 + 0.2)^3 \]

3. (20 Points) Consider the conservative vector field \( \mathbf{F} = (yz \ln(x) + yz) \mathbf{i} + (xz \ln(x)) \mathbf{j} + (xy \ln(x)) \mathbf{k} \).

(a) By direct calculation, determine the integral required to determine the flow along the path \( \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k} \) for \(1 \leq t \leq 2\). Be sure to simplify the integral as much as possible. You are either going to be able to evaluate the integral, or not. If you can, do so. If not, write “I’m going back to Calc II.”

**Solution:** Note that \( \mathbf{v}(t) = (1, 2t, 3t^2) \). It follows that

\[
W = \int_{1}^{2} \mathbf{F} \cdot \mathbf{v} \, dt
\]

\[
= \int_{1}^{2} (t^5 \ln(t) + t^5 + t^5 \ln(t)) \cdot (1, 2t, 3t^2) \, dt
\]

\[
= \int_{1}^{2} 6t^5 \ln(t) + t^5 \ln(t) \, dt
\]

\[
= \left[ \frac{t^6}{6} \ln(t) + \frac{t^6}{12} \right]_{1}^{2}
\]

\[
= \frac{64}{6} - \frac{1}{6} + \frac{1}{6} (64 \ln(64) - 63)
\]

\[
= \frac{64}{6} \ln(64)
\]

\[ = 64 \ln(2) \]

(b) Determine the potential function \( f \) of the vector field \( \mathbf{F} \), and use it to verify (or, if you are “going back to Calc II,” use it to determine) your result in part (a).

**Solution:** Note that at \( t = 1.1 \) the position is \((2, 4, 8)\). Thus,

\[
W = f(2, 4, 8) - f(1, 1, 1)
\]

\[
= 2 \cdot 4 \cdot 8 \cdot \ln(2)
\]

\[ = 64 \ln(2) \]
(c) Let $P$ be the point on the path corresponding to $t = 1$. If one were to continue along the path $\mathbf{r}(t)$ from $P$ for $1 \leq t \leq 1.1$, use Calculus III concepts to estimate the change in the value of the potential function, $\Delta f$.

**Solution:** Let $x_0 = (1, 1, 1)$. The change in the value of $f$ can be approximated with

$$\Delta f \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \frac{\partial f}{\partial z} \Delta z$$

where $\Delta x = 0.1, \Delta y = 0.21, \Delta z = 0.331$. Then it follows that

$$\left. \frac{\partial f}{\partial x} \right|_{x_0} = (yz \ln(x) + yz)\big|_{x_0} = 1$$

$$\left. \frac{\partial f}{\partial y} \right|_{x_0} = 0$$

$$\left. \frac{\partial f}{\partial z} \right|_{x_0} = 0$$

Thus,

$$\Delta f \approx 0.1$$

(d) Instead, if one were to continue along the path $\mathbf{r}(t)$ from $P$ for $\Delta s = 0.1$, where $s$ is arc length, use Calculus III concepts to estimate the change in the value of the potential function, $\Delta f$.

**Solution:** The change in the value of $f$ is now given by

$$\Delta f \approx \frac{df}{ds} \Delta s$$

where

$$\frac{df}{ds} = \frac{df}{dt} \frac{1}{|\mathbf{v}|}$$

and

$$|\mathbf{v}| = \sqrt{14}$$

Thus,

$$\Delta f \approx \frac{0.1}{\sqrt{14}}$$

4. (20 Points) Consider the vector field $\mathbf{F} = y^2 z \mathbf{i} + x^2 z \mathbf{j} + yz \mathbf{k}$. Let $S$ denote the portion of the ellipsoid described by $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$, for $z \geq 0$. Your job is to determine the value of $I = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma$, however a direct calculation will prove very tedious. There are two alternate ways to determine the value of $I$. For parts (a) and (b) below, perform these alternate calculations, and clearly explain your reasoning.

(a) Method 1. Be sure to explain your reasoning.

**Solution:** Consider the closed path around the bottom of the ellipsoid. The parametrization is given by

$$\mathbf{r}(t) = \langle a \cos(t), b \sin(t), 0 \rangle, \quad t \in [0, 2\pi]$$

Then $\mathbf{F}(\mathbf{r}(t)) = \langle 0, 0, 0 \rangle$. Using Stoke’s Theorem it follows that

$$I = \oint \mathbf{F} \cdot \mathbf{T} \, ds$$

$$= \oint 0 \, ds$$

$$= 0$$
(b) Method 2. Be sure to explain your reasoning.

Solution: Let

\[ G = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z & x^2yz & z^2y \\ \end{vmatrix} \]

\[ = (z - x^2)i + (y^2 - 0)j + (2xz - 2yz)k \]

Then it follows that

\[ \nabla \cdot G = \frac{\partial}{\partial x}(z - x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(2xz - 2yz) \]

\[ = -2x + 2y + 2x - 2y \]

\[ = 0 \]

By the Divergence Theorem,

\[ \int \int_S G \cdot n \, d\sigma = \int \int \int_D \nabla \cdot G \, dV \]

\[ = \int \int \int_D 0 \, dV \]

\[ = 0 \]

But to calculate \( I \) the flux through the bottom of the ellipsoid needs to be subtracted. The outward flux through the bottom is given by

\[ \oint F \cdot n \, ds = \oint \langle y^2z, x^2z, yz \rangle \cdot \langle 0, 0, -1 \rangle \, ds \]

\[ = \int 0 \, ds \]

\[ = 0 \]

Thus,

\[ I = 0 - 0 = 0 \]

5. (20 points) The volume of a solid object is described by the integral \( V = \int_0^{2\pi} \int_0^1 \int_r \sqrt{r} \, dz \, dr \, d\theta \).

(a) Now, consider the vector field \( F = yz \, i - xz \, j + z^4 \, k \). Find the total outward flux of \( F \) across the entire surface of the solid.

Solution: Observe that the divergence of \( F \) is

\[ \nabla \cdot F = \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(z^4) \]

\[ = 4z^3 \]

Using the Divergence Theorem it follows that

\[ \int \int \int 4z^3 \, dV = \int_0^{2\pi} \int_0^1 \int_r \sqrt{r} \, dz \, dr \, d\theta \]

\[ = 2\pi \int_0^1 r z^4 \sqrt{r} \, dr \]

\[ = 2\pi \int_0^1 r^3 - r^5 \, dr \]

\[ = 2\pi \left[ \frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 \]

\[ = 2\pi \left( \frac{1}{4} - \frac{1}{6} \right) \]

\[ = \frac{\pi}{6} \]
(b) If it is possible to verify your calculation in part (a) using a different method, do so. Be sure to specifically name any theorems you use.

**Solution:** To calculate the total outward flux across the entire surface we will calculate the outward flux across the “top” and “bottom” surfaces and sum them. The flux through the bottom surface \((r = z)\) is given by

\[
\int \int \mathbf{F} \cdot \frac{\nabla \mathbf{G}}{|\nabla \mathbf{G}|} \frac{|\nabla \mathbf{g} \cdot \mathbf{p}|}{|\nabla \mathbf{g} \cdot \mathbf{p}|} dA
\]

where \(g = \sqrt{x^2 + y^2} - z\). It follows that

\[
\nabla \mathbf{G} = \left\langle \frac{2x}{\sqrt{x^2 + y^2}}, \frac{2y}{\sqrt{x^2 + y^2}}, -1 \right\rangle
\]

\[
\nabla \mathbf{G} \cdot \mathbf{F} = -z^4
\]

Observe that \(|\nabla \mathbf{g} \cdot \mathbf{p}| = 1\). Therefore, the flux through the bottom surface is

\[
\int_0^{2\pi} \int_0^1 -z^4r \, dr \, d\theta = 2\pi \int_0^1 -r^5 \, dr = 2\pi \left[ -\frac{r^6}{6} \right]_0^1 = -\frac{\pi}{3}
\]

For the flux through the top surface \((z = \sqrt{r})\), let \(g = z - (x^2 + y^2)^{1/4}\). Then it follows that the flux is given by

\[
\int_0^{2\pi} \int_0^1 z^4r \, dr \, d\theta = 2\pi \int_0^1 r^3 \, dr = 2\pi \frac{1}{4} = \frac{\pi}{2}
\]

Thus, the total flux is

\[
\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}
\]

ENJOY YOUR SUMMER!
Projections and distances \[ \text{proj}_A \mathbf{B} = \left( \frac{\mathbf{A} \cdot \mathbf{B}}{\mathbf{A} \cdot \mathbf{A}} \right) \mathbf{A} \quad d = \frac{|\mathbf{P}_S \times \mathbf{v}|}{|\mathbf{v}|} \quad d = \frac{\mathbf{P}_S \cdot \mathbf{n}}{|\mathbf{n}|} \]

Arc length, frenet formulas, and tangential and normal acceleration components
\[
\frac{ds}{dt} = |\mathbf{v}|, \quad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|}, \quad \mathbf{N} = \frac{dT/ds}{|dT/ds|} = \frac{dT}{|dT|}, \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}
\]

\[
\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}, \quad \frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}, \quad \kappa = \frac{|d\mathbf{T}|}{|\mathbf{T}|}, \quad \tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}
\]

The Second Derivative Test
Suppose \( f(x, y) \) and its first and second partial derivatives are continuous in a disk centered at \((a, b)\), and \( f_x(a, b) = f_y(a, b) = 0 \).

Let \( f_{xx}f_{yy} - f_{xy}^2 \).
\[
1. \text{ If } D > 0 \text{ and } f_{xx} < 0 \text{ at } (a, b), \text{ then } f \text{ has a local maximum at } (a, b).
2. \text{ If } D > 0 \text{ and } f_{xx} > 0 \text{ at } (a, b), \text{ then } f \text{ has a local minimum at } (a, b).
3. \text{ If } D < 0 \text{ at } (a, b), \text{ then } f \text{ has a saddle point at } (a, b).
4. \text{ If } D = 0 \text{ at } (a, b), \text{ then the test is inconclusive.}
\]

Directional derivative, discriminant, and Lagrange multipliers
\[
\frac{df}{ds} = (\nabla f) \cdot \mathbf{u}, \quad f_{xx}f_{yy} - (f_{xy})^2, \quad \nabla f = \lambda \nabla g, \quad g = 0
\]

Taylor’s formula (at the point \((x_0, y_0)\))
\[
f(x, y) = f(x_0, y_0) + \left[ (x - x_0)f_x(x_0, y_0) + (y - y_0)f_y(x_0, y_0) \right]
+ \frac{1}{2!} \left[ (x - x_0)^2f_{xx}(x_0, y_0) + 2(x - x_0)(y - y_0)f_{xy}(x_0, y_0) + (y - y_0)^2f_{yy}(x_0, y_0) \right]
+ \frac{1}{3!} \left[ (x - x_0)^3f_{xxx}(x_0, y_0) + 3(x - x_0)^2(y - y_0)f_{xxy}(x_0, y_0)
+ 3(x - x_0)(y - y_0)^2f_{xyy}(x_0, y_0) + (y - y_0)^3f_{yyy}(x_0, y_0) \right] + \cdots
\]

Linear approximation error
\[
|E(x, y)| \leq \frac{M}{2!} \left| \sum_{i=2}^n \frac{f_{ji}(x_0, y_0)}{i!} \right|^2, \quad \text{ where } \max \left\{ |f_{xx}|, |f_{xy}|, |f_{yy}| \right\} \leq M
\]

Polar coordinates \( x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2, \quad dA = dx \, dy = r \, dr \, d\theta \)

Cylindrical and spherical coordinates

<table>
<thead>
<tr>
<th>Cylindrical to Rectangular</th>
<th>Spherical to Cylindrical</th>
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<tbody>
<tr>
<td>( x = r \cos \theta )</td>
<td>( r = \rho \sin \phi )</td>
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<tr>
<td>( y = r \sin \theta )</td>
<td>( z = \rho \cos \phi )</td>
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<tr>
<td>( z = z )</td>
<td>( y = \rho \sin \phi \cos \theta )</td>
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<td>( \theta = \theta )</td>
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<tr>
<td></td>
<td>( z = \rho \cos \phi )</td>
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</tbody>
</table>

\[
dV = dx \, dy \, dz = dz \, r \, d\theta \, dr = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

Substitutions in multiple integrals
\[
\int_R \int_D f(x, y) \, dx \, dy = \int_D \int_C f(x(u, v), y(u, v)) \, |J(u, v)| \, du \, dv \quad \text{where} \quad J(u, v) = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right|
\]

Mass, moments, and center of mass

\[
\text{Mass } M = \int_R \delta \, dA \quad \text{Moments } M_x = \int_R y \delta \, dA, \quad M_y = \int_R x \delta \, dA \quad \text{Center of mass } \bar{x} = M_y / M, \quad \bar{y} = M_x / M
\]

Green’s Theorem in the \( x-y \) plane (The curve \( C \) is traversed counterclockwise, and \( \mathbf{F} = M \mathbf{i} + N \mathbf{j} \))
\[
\text{Circulation } = \oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_R \left( \nabla \times \mathbf{F} \right) \cdot \mathbf{k} \, dA = \int_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA
\]
\[
\text{Outward Flux } = \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \int_R \left( \nabla \cdot \mathbf{F} \right) \, dA
\]

Surface area of level surface \( g(x, y, z) = c \)
\[
S = \int_S \, d\sigma = \int_R \left( \frac{\nabla g}{|\nabla g|} \right) \, dA
\]

Stoke’s Theorem \[
\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma
\]

Divergence Theorem of Gauss \[
\int_S \mathbf{F} \cdot \mathbf{n} \, d\sigma = \int_D \nabla \cdot \mathbf{F} \, dV
\]

Fond memories \[
\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}
\]