1. Solve the initial value problem \( y' = 2te^{-t^2} \csc(3y) \), \( y(0) = 2 \).

**Sol:** The DE is separable:

\[
y' = 2te^{-t^2} \csc(3y)
\]

\[\Rightarrow \sin(3y) \, dy = 2te^{-t^2} \, dt\]

\[\Rightarrow \int \sin(3y) \, dy = \int 2te^{-t^2} \, dt + C\]

\[\Rightarrow \frac{-\cos(3y)}{3} = -e^{-t^2} + C\]

\[\Rightarrow \cos(3y) = 3e^{-t^2} + C\]

Now use initial condition \( y(0) = 2 \) to solve for \( C \):

\[\cos(6) = 3 + C \Rightarrow C = \cos(6) - 3\]

Therefore, our solution is

\[\cos(3y) = 3e^{-t^2} + \cos(6) - 3\]

or

\[y = \frac{\arccos(3e^{-t^2} + \cos(6) - 3)}{3}\]

2. Consider the DE \( y' = \frac{2y}{t} \).

(a) State Picard’s existence and uniqueness theorem regarding solutions to the IVP:

\[y' = f(t, y), \quad y(t_0) = y_0\]

**Sol:** The theorem states: If \( f(t, y) \) is continuous on the region

\[R = \{(t, y) \mid t \in (a, b), \ y \in (c, d)\}\]

and \((t_0, y_0) \in R\), then there exists a number \( h > 0 \) such that the IVP has a solution for \( t \in (t_0 - h, t_0 + h) \). Moreover, if \( f_y(t, y) \) is also continuous in \( R \), the solution is unique.

(b) What does Picard’s Theorem tell us about the solutions to the IVP with following initial conditions?

**Sol:** In this case, we have \( f(t, y) = \frac{2y}{t} \) and \( f_y(t, y) = \frac{2}{t} \). Both are continuous for all values of \( t \) and \( y \) except when \( t = 0 \). By Picard’s Theorem, a unique solution exists for all initial conditions \( y(t_0) = y_0 \) with \( t_0 \neq 0 \).
• $y(1) = 1$: a unique solution to the IVP exists in some neighborhood of the initial point.
• $y(0) = 0$: conditions of Picard’s Theorem are not satisfied since $f$ is not continuous at $t = 0$. The theorem does not tell us anything about our solutions.
• $y(0) = 1$: same as (ii).

c) Determine any solutions (if they exist) to each of the IVPs above.

**Sol:** By separation of variables:

$$\frac{dy}{dt} = \frac{2y}{t}$$

$$\implies \frac{dy}{2y} = \frac{dt}{t}$$

$$\implies \int \frac{dy}{2y} = \int \frac{dt}{t} + C$$

$$\implies \frac{1}{2} \log |y| = \log |t| + C$$

$$\implies y(t) = Ct^2.$$  

• $y(1) = 1$: with initial condition we have $C = 1$. Thus the unique solution to the IVP is $y(t) = t^2$.
• $y(0) = 0$: the equilibrium solution $y(t) = 0$ satisfies the IVP with $y(0) = 0$. Additionally, all solutions of the form $y(t) = Ct^2$ pass through the origin for any constant $C$. Thus the IVP with $y(0) = 0$ has infinitely many solutions.
• $y(0) = 1$: in the case there are no solutions since the initial condition can never be satisfied with $y(t) = Ct^2$.

3. (a) Find a rectangle on which Picard’s conditions apply to the differential equation

$$y' = \frac{y - t}{y + t}, \quad y(0) = -1$$

**Sol:** We have $f(t, y) = \frac{y - t}{y + t}$ and $f_y(t, y) = \frac{2t}{(y + t)^2}$. Both are continuous except on the line $y = -t$. So we can make any rectangle that contains the initial point $(0, -1)$ and which does not bump into the line $y = -t$. For example $R = [-0.25, 0.25] \times [-1.5, -0.5]$ works.

(b) The direction field for this DE is shown in Figure 1. What goes wrong on the line $y = -t$?

**Sol:** The slope would be infinite on that line, which does not make sense physically because $y$ cannot have an infinity rate of change.
(c) What happens if we try to solve the initial value problem numerically for \( t \) in the interval \([0, 1]\)? Try this using Euler’s method with step size \( h = 0.2 \). Does this look like a reasonable solution?

Sol: When we try to solve the problem numerically, we can “jump” over the discontinuity and continue on the other side. However, this is clearly not a reasonable approximation to the real behavior of the system. The following table describes the results from Euler’s method, where \( \hat{y} \) is our approximation of \( y \).

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<th>( t )</th>
<th>( \hat{y} )</th>
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<tr>
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</tbody>
</table>

4. Categorize the following differential equations according to their order, as linear or nonlinear, as homogeneous or non-homogeneous, and as having constant or variable coefficients.

(a) \( y'' + y' = 3t \sin y \)

(b) \( 3y''' - t^3y' + e^t y = 0 \)

(c) \( y'' + (\log t)y' = e^t \)

(d) \( y^{(4)} + 7y''' + \frac{7}{2}y'' + y' + 2y = \cos t \)

Sol:

(a) 2nd-order, nonlinear

(b) 3rd-order, linear, homogeneous, variable coefficients

(c) 2nd-order, linear, non-homogeneous, variable coefficients

(d) 4th-order, linear, non-homogeneous, constant coefficients.