Exam 1 Solutions

February 10, 2015

1 Question 1

1.1 Part (A)

To find equilibrium solutions, set $P(t) = C \implies \frac{dP}{dt} = 0$. This implies:

$$\frac{dP}{dt} = P(1 - P) - 2P = P - P^2 - 2P = -P - P^2 = -P(1 + P) = 0$$

The equilibrium solutions are thus $P(t) = 0$ and $P(t) = -1$.

1.2 Part (B)

We expand and use the method of partial fractions.

$$\frac{dP}{dt} = P(1 - P) - 2P = P - P^2 - 2P = -P - P^2 = -P(1 + P)$$

Separating variables we get:

$$\frac{dP}{P(1 + P)} = -dt$$

where

$$\frac{1}{P(1 + P)} = \frac{A}{P} + \frac{B}{1 + P} \implies 1 = A(1 + P) + BP$$

Set $P = 0$ to have $A + 0 = 1 \implies A = 1$. Set $P = -1$ to have $1 = -B \implies B = -1$. Thus:

$$\frac{dP}{P(1 + P)} = \frac{dP}{P} - \frac{dP}{P + 1} = -dt$$

Integrating we get:

$$\log|P| - \log|P + 1| = -t + C_1 \implies \log\left|\frac{P}{P + 1}\right| = -t + C_1 \implies \frac{P(t)}{P(t) + 1} = Ce^{-t}$$

Factoring and simplifying yields:

$$P(t) \left[1 - Ce^{-t}\right] = Ce^{-t} \implies P(t) = \frac{Ce^{-t}}{1 - Ce^{-t}} = \frac{C}{e^t - C}$$
1.3 Part (C)

We plug in the initial condition and solve for $C$:

$$P(t) = \frac{C}{e^t - C} \implies P(0) = \frac{C}{1 - C} = 1 \implies C = \frac{1}{2} \implies P(t) = \frac{1}{2e^t - 1}$$
Problem 1: (30 points) Consider the differential equation

\[ \frac{dP}{dt} = P(1 - P) - 2P \]

(a) (5 points) Find all equilibrium solutions.
(b) (20 points) Find the general solution \( P(t) \).
(c) (5 points) Solve the IVP where \( P(0) = 1 \).

Solution: Part (A)
We expand and use the method of partial fractions.

\[ \frac{dP}{dt} = P(1 - P) - 2P = P - P^2 - 2P = -P - P^2 = -P(1 + P) \]

Separating variables we get:

\[ \frac{dP}{P(1 + P)} = -dt \]

where

\[ \frac{1}{P(1 + P)} = \frac{A}{P} + \frac{B}{1 + P} \implies 1 = A(1 + P) + BP \]

Set \( P = 0 \) to have \( A + 0 = 1 \implies A = 1 \). Set \( P = -1 \) to have \( 1 = -B \implies B = -1 \). Thus:

\[ \frac{dP}{P(1 + P)} = \frac{dP}{P} - \frac{dP}{P + 1} = -dt \]

Integrating we get:

\[ \log |P| - \log |P + 1| = -t + C_1 \implies \log \left| \frac{P}{P + 1} \right| = -t + C_1 \implies \frac{P(t)}{P(t) + 1} = Ce^{-t} \]

Factoring and simplifying yields:

\[ P(t) \left[ 1 - Ce^{-t} \right] = Ce^{-t} \implies P(t) = \frac{Ce^{-t}}{1 - Ce^{-t}} = \frac{C}{e^{t} - C} \]

Part (B)
In standard form, the ODE is: \( \frac{dP(t)}{dt} = -P(t) - P(t)^2 = f(P, t) \). The function \( f(P, t) \) is continuous at and in a rectangle around \( (t, P) = (0, 1) \). Also, the partial derivative \( \frac{\partial}{\partial P} f(P, t) = -1 - 2P(t) \) is also continuous at and in a rectangle around \( (t, P) = (0, 1) \). Hence, Picard’s theorem guarantees the existence of a unique solution to the IVP.

Part (C)
We plug in the initial condition and solve for \( C \):

\[ P(t) = \frac{C}{e^{t} - C} \implies P(0) = \frac{C}{1 - C} = 1 \implies C = \frac{1}{2} \implies P(t) = \frac{1}{2e^{t} - 1} \]

Problem 2: (30 points) Answer the following questions.

(a) (15 points) Solve the IVP

\[ ty' + (1 - t)y = 1, \quad y(1) = e - 1 \]

using the method of integrating factors.
(b) (5 points) If the initial condition is changed to \( y(t_0) = 4 \) for which values of \( t_0 \) does Picard’s theorem guarantee a unique solution?

(c) (10 points) Consider the ODE

\[
\frac{dy}{dt} + (\sin t)y = f(t)
\]

Using the two stage Euler-Lagrange method (i.e., method of variation of parameters) show that the particular solution has the form

\[
y_p(t) = v(t)e^{\cos t}
\]

where \( v(t) \) satisfies

\[
\frac{dv}{dt} = f(t)e^{-\cos t}
\]

**Solution:** Integrating factors

(a) Identify normalized form of the ODE as

\[
y' + \left(\frac{1}{t} - 1\right)y = \frac{1}{t}
\]

and see that \( p(t) = \left(\frac{1}{t} - 1\right) \). Find the integrating factor

\[
\mu(t) = \exp \int p(t)dt = \exp \int \left(\frac{1}{t} - 1\right) dt = te^{-t}
\]

Multiplying the ODE by integrating factor gives

\[
\frac{d}{dt} \left(te^{-t}y(t)\right) = e^{-t}
\]

Integrate for general solution

\[
y(t) = \frac{1}{t} \left(Ce^{t} - 1\right)
\]

Solve IVP

\[ e - 1 = Ce - 1 \quad \Rightarrow \quad C = 1. \]

to find particular solution

\[
y_p(t) = \frac{1}{t} \left(e^{t} - 1\right).
\]

(b) Given

\[
y' = \frac{1}{t} - \left(\frac{1}{t} - 1\right)y
\]

Identify rate function

\[
f(t, y) = \frac{1}{t} - \left(\frac{1}{t} - 1\right)y, \quad f_y(t, y) = -\left(\frac{1}{t} - 1\right)
\]

Both functions are well-defined provide \( t \neq 0 \). Thus a uniques solution exist \( \forall \, t \neq 0 \).

(c) Substitute ansatz into the ODE to find

\[
\left(e^{\cos t}\frac{dv}{dt} - \sin t \, e^{\cos t}v\right) + \sin t \left(v(t)e^{\cos t}\right) = f(t)
\]

Then

\[
e^{\cos t}\frac{dv}{dt} = f(t)
\]

and answer follows immediately.

**Problem 3:** (30 points) Answer the following questions TRUE or FALSE. You DO NOT need to justify your answer.

(a) All isoclines of an autonomous first-order diffeq must be horizontal lines.
(b) For the IVP $y' = t, \ y(0) = 1$ the absolute value of the approximation error in using Euler’s method with step size $h = 1$ after one step is $1/2$.

(c) All first-order autonomous diff eqs must have at least one equilibrium solution.

(d) Picard’s Theorem guarantees that the IVP $y' = y^{1/3}, \ y(0) = 0$ has the unique equilibrium solution $y(t) = 0$.

(e) All equilibrium points of $\frac{dy}{dt} = -y^2(4 - y^2)$ are stable.

(f) The diff eq $y^{(4)} = \sec^2(t) + \ln(e^{ty})$ is linear.

**Solution:**

(a) True. If $y' = f(y)$, then solving $y' = c$ can only yield constant $y$-values as solutions.

(b) True. Note the IVP has solution $y(t) = t^2/2 + 1 \implies y(1) = 3/2$. With $t_0 = 0, \ y_0 = 1$ and $f(t, y) = t$, one application of Euler’s method gives $y_1 = y_0 + h f(t_0, y_0) = 1 + 1(0) = 1$ so that $|y_1 - y(1)| = 1/2$.

(c) False. $y' = 2$ has none.

(d) False. Here $y' = f(t, y)$ with $f(t, y) = y^{1/3} \implies f_y(t, y) = (1/3)y^{-2/3}$ which is not continuous on any rectangle containing $(0,0)$. In fact, $y(t) = (2t/3)^{3/2}$ is another solution.

(e) False. $y = -2$ is unstable.

(f) True. $y^{(4)}$ indicates a fourth derivative, and note that $\ln(e^{ty}) = ty$. 
Problem 4: (30 points) Alice is a graduate student in chemistry. In her lab she finds a tank that contains 10 gallons of chemical solution, which she labels as solution A. The chemical concentration of solution A is measured to be 2 pounds per gallon. She also finds plenty of chemical solution with chemical concentration 4 pounds per gallon, which she labels as solution B. To achieve the chemical concentration that she desires, she injects solution B into solution A at a flow rate of 3 gallons per hour and drains the well-mixed mixture out at a flow rate of 3 gallons per hour.

(a) (8 points) Set up the initial value problem for the amount of the chemical in the mixture as a function of time.

(b) (12 points) Solve the initial value problem.

(c) (6 points) Suppose that instead of draining the mixture at 3 gallons per hour, Alice evaporates the mixture at 3 gallons per hour. In this process, water escapes from the mixture while the chemical remains. In this case, what is the amount of the chemical in the mixture as a function of time?

(d) (4 points) To reach a chemical concentration of 3 pounds per gallon, is it more time efficient to evaporate the mixture or to drain the mixture? You do not need to justify your answer.

Solution:

(a) Let the chemical amount as a function of time be \( x(t) \). Then the differential equation is

\[
x'(t) = \text{(rate in)} - \text{(rate out)} = 4 \text{ (lb/gal)} \times 3 \text{ (gal/hr)} - \frac{x}{10} \text{ (lb/gal)} \times 3 \text{ (gal/hr)} = 12 - \frac{3x(t)}{10} \text{ (lb/hr)}.
\]

The initial condition is

(1) \( x(0) = 2 \text{ (lb/gal)} \times 10 \text{ (gal)} = 20 \text{ (lb)} \).

(b) To solve the differential equation, we rewrite it in standard form

(2) \[ x'(t) + \frac{3}{10} x(t) = 12. \]

The homogeneous solution is

(3) \[ x_h'(t) + \frac{3}{10} x_h(t) = 0 \quad \Rightarrow \quad x_h(t) = ce^{-\frac{3}{10}t}. \]

A particular solution is, by inspection

(4) \[ x_p(t) = 40. \]

Therefore the general solution is

(5) \[ x(t) = x_h(t) + x_p(t) = ce^{-\frac{3}{10}t} + 40. \]

Using the initial condition, we have

(6) \[ c + 40 = 20 \quad \Rightarrow \quad c = -20. \]

Hence the solution to the initial value problem is

(7) \[ x(t) = -20e^{-\frac{3}{10}t} + 40. \]

(c) In this case the concentration of the outgoing flow is 0, so the initial value problem becomes

(8) \[ x'(t) = 12, \quad x(0) = 20. \]

The solution is then

(9) \[ x(t) = 20 + 12t. \]

(d) It is more time efficient to evaporate the mixture. This is because the rate at which the chemical amount increases is always greater for evaporation than drainage.
Problem 5: (30 points) Consider the following first order ordinary differential equations:

\begin{align*}
(1) \quad y' &= \frac{\sin(4t)}{y}, \\
(2) \quad y' &= (y + 1)(y - 2), \\
(3) \quad y' &= -2y(2 - y), \\
(4) \quad y' &= (y - 2)\sqrt{y + 1}, \\
(5) \quad y' &= \frac{2(y - 2)}{y + 1}, \\
(6) \quad y' &= \frac{t}{y},
\end{align*}

(a) (4 points) Which of the above differential equations are linear? Which of the above differential equations are autonomous?

(b) (16 points) Match direction fields (A)-(D) with its corresponding differential equations. (Note that there are more equations than direction fields, so two equations have no corresponding direction fields.)

(c) (10 points) Where appropriate, identify the equilibria shown in \textbf{EACH GRAPH} and give their stability.
Solution:

(a)  
- Linear: None.
- Autonomous: (2), (3), (4), (5).

(b)-(c)  
- Graph (A): (3); equilibria: \( y = 0 \): stable, \( y = 2 \): unstable.
- Graph (B): (2); equilibria: \( y = -1 \): stable, \( y = 2 \): unstable.
- Graph (C): (1); equilibria: None.
- Graph (D): (5); equilibria: \( y = 2 \): unstable.