Problem #1 (10 points): Represent the function
\[ \frac{e^z}{z(z^2 + 1)} \]
As a Laurent series in the domain \(|z| < 1\)

Problem #2 (10 points): Prove that
\[ f(z) = \int_0^1 \frac{\cos t}{t-z} \, dt + \int_2^3 \frac{\sin t}{t-z} \, dt \]
admits a Laurent expansion in the ring \(1 < |z| < 2\) and compute this expansion.

Problem #3 (18 points): Find all the singular points (be sure to check \(z_\infty\)) for the following functions:

(a) \(\frac{e^{z^2} - 1}{z^2}\),
(b) \(\frac{1}{z - \sin z}\),
(c) \(e^{\tan z}\),
(d) \(\frac{z^3}{z^2 + z + 1}\),
(e) \(\log(1 + z^{1/2})\),
(f) \(\frac{1}{(\sin z + \cos z - 1)^2}\).

Then classify them as either isolated or non-isolated and by type (such as removable, pole of order \(N\) and strength \(c_{-N}\), essential, branch point, cluster point, or natural barrier).

Problem #4 (15 points): Evaluate the integral \(\oint_C f(z) \, dz\), where \(C\) is a unit circle centered at the origin, for the following functions:

(a) \(\frac{g(z)}{z - \omega}\), where \(g(z)\) is entire,
(b) \(\frac{z}{z^2 - \omega^2}\),
(c) \(z e^{1/z^2}\),
(d) \(\cot z\),
(e) \(\frac{1}{8z^3 + 1}\).

Problem #5 (16 points): Determine if the following functions are meromorphic. And if they’re meromorphic, determine the order, strength, and location of all their poles.

(a) \(\frac{z}{z^4 + 2}\),
(b) \(\frac{z}{\sin^2 z}\),
(c) \(\frac{e^z - 1 - z}{z^4}\),
(d) \(\tan z\).

Problem #6 (8 points): Assume that \(z_0\) is a zero (resp. a pole) of order \(N\) of the function \(f\), and let \(M \in \mathbb{N}\). Show that \(z_0\) is a zero (resp. a pole) of order \(NM\) of \(f^M\).

Problem #7 (10 points): What are the singularities of \(h(z) = \frac{z^3(z - 1)^6}{\sin^5(\pi z)}\)?

Hint: \(z = 0\) is pole of order 2, \(z = 1\) is a removable singularity which is a zero of order 1, and all other integers are poles of order 5.

Problem #8 (13 points): Let \(a\) be an isolated singular point of \(f\). Prove the following:

(a) If \(\lim_{z \to a} (z-a)f(z) = 0\), then \(a\) is a removable singularity.
(b) If there exists an \(m \in \mathbb{N}\), \(m > 0\) such that \(\lim_{z \to a} (z-a)^m f(z) = c_{-m} \neq 0\), then \(a\) is a pole of order \(m\) with strength \(c_{-m}\).

Extra-Credit Problem #9 (10 points): Find the poles and the zeros of the function
\[ f(z) = \frac{\cos z(z^2 - 1)^3 \sin(z^2) \sin(\pi z)}{(e^z - 1)(z^2 + 1)} \]