Problem #1 (25 points): Evaluate the integral
\[ I = \frac{1}{2\pi i} \oint_C f(z) \, dz, \]
where \( C \) is the unit circle centered at the origin, for the following \( f(z) \):

(a) \( f(z) = \frac{z^2 + 1}{z^2 - a^2}, \quad a^2 < 1 \)
(b) \( f(z) = z \cos \frac{1}{z + 1} \)
(c) \( f(z) = z^2 e^{-1/z} \)

Problem #2 (12 points): For complex \( a \) with \(|a| < 1\), show that
\[ I(a) = \int_0^{2\pi} \frac{d\theta}{1 - 2a \cos \theta + a^2} = \frac{2\pi}{1 - a^2}. \]
Now find the result when \(|a| > 1\)?

Problem #3 (24 points): Let \( C \) be the unit circle centered at the origin. Evaluate the integral
\[ I = \frac{1}{2\pi i} \oint_C f(z) \, dz, \]
for the following \( f(z) \) in two ways: (i) enclosing the singular points inside \( C \) and (ii) enclosing the singular points outside \( C \) (by including the point at infinity). Do you get the same result in both cases?

(a) \( f(z) = \frac{z^2 + 1}{z^2 - a^2}, \quad a^2 < 1 \)
(b) \( f(z) = z \cos \frac{1}{z + 1} \)
(c) \( f(z) = z^2 e^{-1/z} \)
(d) \( f(z) = \frac{z - z^{-1}}{z(2z + (2z)^{-1})} \)

Problem #4 (24 points): What type of singularity do the following functions have at \( z = \infty \)?

(a) \( z^m, \quad m \in \mathbb{N} \)
(b) \( z^{1/3} \)
(c) \( (z^2 + a^2)^{1/2}, \quad a^2 > 0 \)
(d) \( \log(z^2 + a^2), \quad a^2 > 0 \)
(e) \( e^z \)
(f) \( z^2 \sin z^{-1} \)
(g) \( \sin^{-1} z \)
(h) \( \log(1 - e^{1/z}) \)

Problem #5 (15 points): Assume that \( f \) and \( g \) are analytic outside a circle \( C_R \) of radius \( R \) centered at the origin and
\[ \lim_{|z| \to \infty} f(z) = C_1 \quad \text{and} \quad \lim_{|z| \to \infty} zg(z) = C_2, \]
where \( C_1 \) and \( C_2 \) are constants. Show that
\[ \frac{1}{2\pi i} \oint_{C_R} g(z) e^{f(z)} \, dz = C_2 e^{C_1}. \]

Extra-Credit Problem #6 (15 points): Find
\[ I(a) = -i\pi \oint_{C_a} \frac{e^z}{z(z^2 + \pi^2)} \, dz \]
for \(-\infty < a < \infty \) and where \( C_a \) is the rectangle with corners \(-1 + ia, -1 + i(a + 4), 1 + i(a + 4), \) and \( 1 + ia \).