Problem #1 (10 points): Use the Taylor series for \((1 + z)^{-1}\) about \(z = 0\) to find the Taylor series of \(\log(1 + z)\) about \(z = 0\) for \(|z| < 1\).

Solution: The Taylor series for \((1 + z)^{-1}\) is just the geometric series

\[
\frac{1}{1 + z} = \sum_{n=0}^{\infty} (-1)^n z^n,
\]

and we know that it converges uniformly in \(|z| < 1\). Since \(\int (1 + z)^{-1} \, dz = \log(1 + z) + c\) and since the above series converges uniformly so we can integrate it term-wise, we find that

\[
\log(1 + z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}.
\]

Moreover, this series will converge uniformly for \(|z| < 1\).

Problem #2 (12 points): Use the Taylor series for \((1 - z)^{-1}\) about \(z = 0\) (which converges when \(|z| < 1\)) to find a series representation of \((1 - z)^{-1}\) that converges when \(|z| > 1\). Hint: \((1 - z)^{-1} = -[z(1 - 1/z)]^{-1}\).

Solution: We know that

\[
\frac{1}{1 - \eta} = \sum_{k=0}^{\infty} \eta^k,
\]

when \(|\eta| < 1\). Using the hint,

\[
\frac{1}{1 - z} = -\frac{1}{z} \left( \frac{1}{1 - 1/z} \right) = -\frac{1}{z} \sum_{k=0}^{\infty} \frac{1}{z^k} = -\sum_{k=0}^{\infty} \frac{1}{z^{k+1}}
\]

converges when \(|1/z| < 1\) or \(|z| > 1\).

Problem #3 (12 points): Expand

\[
f(z) = \frac{1}{1 + z^2}
\]

about \(z = 0\) in

(a) a Taylor series for \(|z| < 1\) and (b) a Laurent series for \(|z| > 1\).

Solution:

(a) The Taylor series for \(|z| < 1\) is

\[
\frac{1}{1 + z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}.
\]

(b) The Laurent series for \(|z| > 1\) is

\[
\frac{1}{1 + z^2} = \frac{1}{z^2} \left( \frac{1}{1 + 1/z^2} \right) = \sum_{n=0}^{\infty} (-1)^n.
\]

Problem #4 (18 points): Expand

\[
f(z) = \frac{z}{(z-2)(z+i)}
\]

in a Laurent series about \(z = 0\) in the following regions:

(a) \(|z| < 1\), (b) \(1 < |z| < 2\), (c) \(|z| > 2\)

Solution: Using partial fractions, we see that

\[
\frac{z}{(z-2)(z+i)} = \frac{4/5 - 2i/5}{z-2} + \frac{1/5 + 2i/5}{z+i}.
\]

(a) For \(|z| < 1\),

\[
f(z) = -\frac{2/5 + i/5}{1-z/2} + \frac{2/5 - i/5}{1-iz} \sum_{n=0}^{\infty} \left( \frac{1}{2^n} - (i)^n \right) z^n.
\]

(b) For \(1 < |z| < 2\),

\[
f(z) = -\frac{2/5 + i/5}{1-z/2} + \frac{1/5 + 2i/5}{z} \sum_{n=0}^{\infty} \left( \frac{1}{z^n} + (i)^n \right) z^{-n}.
\]

(c) For \(|z| > 2\),

\[
f(z) = \frac{4/5 - 2i/5}{z} \sum_{n=0}^{\infty} \left( \frac{1}{z^n} - (i)^n \right) \frac{1}{z^{n+1}}.
\]

Problem #5 (30 points): Evaluate the integral

\[\oint_C f(z) \, dz\]

where \(C\) is the unit circle centered at the origin for the following \(f(z)\):

(a) \(\frac{e^z}{z^5}\)
Problem #6 (18 points): Let
\[ \exp \left( \frac{t}{2} (z - 1/z) \right) = \sum_{n=\infty}^{\infty} J_n(t) z^n \]
define \( J_n(t) \). Using the definition of Laurent series and the properties of integration, show that
\[ J_n(t) = \frac{1}{2\pi i} \int_{C} e^{-i(n\theta - t\sin\theta)} d\theta \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta - t\sin\theta) d\theta \]
The functions \( J_n(t) \) are called Bessel functions and they're well-known special functions in mathematics and physics.

Solution: Here we just use the definition of the Laurent series:
\[ f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \]
\[ c_n = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{(z - z_0)^{n+1}} dz. \]
Let \( C \) be the unit circle and \( z = e^{i\theta} \):
\[ J_n(t) = \frac{1}{2\pi i} \int_{C} \exp \left\{ \frac{t}{2} (z - 1/z) \right\} \frac{dz}{(z - z_0)^{n+1}} \]
\[ = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \exp \left\{ \frac{t}{2} (e^{i\theta} - e^{-i\theta}) \right\} i e^{i\theta} d\theta \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp \left\{ i t \sin\theta - i n\theta \right\} d\theta \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(t \sin\theta - n\theta) + i \sin(t \sin\theta - n\theta) d\theta \]
\[ = \frac{1}{\pi} \int_{0}^{\pi} \cos(n\theta - t \sin\theta) d\theta, \]
where we used that \( \sin(t \sin\theta - n\theta) \) is an odd function while \( \cos(t \sin\theta - n\theta) \) is an even function.

Extra-Credit Problem #7 (10 points): Use the binomial expansion and Cauchy integral formula to evaluate
\[ \oint_{|z|=1} \left( z + \frac{1}{z} \right)^{2n} \frac{dz}{z} \]
Recall the binomial expansion
\[ (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \]
where
\[ \binom{n}{k} = \frac{n!}{k!(n-k)!}. \]
Use this result to establish the following real integral formula:
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \cos^2 t \cdot \frac{t^{2n}}{2n} d\theta = \frac{(2n)!}{4^n (n!)^2}. \]
**Solution:** Using the binomial expansion, we get

\[
\left(z + \frac{1}{z}\right)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} z^k (1/z)^{2n-k}.
\]

Substituting this into the integral we find

\[
\oint_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = \sum_{k=0}^{2n} \binom{2n}{k} \oint_{|z|=1} z^k (1/z)^{2n-k} \frac{dz}{z} = \int_{|z|=1} \left(2n\right) \frac{dz}{z} = 2\pi i \binom{2n}{n} = 2\pi i \frac{(2n)!}{(n!)^2},
\]

since only $z^{-1}$ term contributes which corresponds to $k = 2n - k$ or $k = n$. For the real integral we find

\[
\frac{1}{2\pi} \int_{0}^{2\pi} (\cos \theta)^{2n} d\theta = \frac{1}{2\pi i} \oint_{|z|=1} \left(e^{i\theta} + e^{-i\theta}\right)^{2n} \frac{ie^{i\theta} d\theta}{e^{i\theta}} = \frac{1}{2\pi i} \oint_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = \frac{(2n)!}{4^n (n!)^2},
\]

where we used the result for the integral in the last equality.