2.2.2: Find general solutions for the equation

\[ \frac{dy}{dt} + 2y = 3e^t \]

Solution: Finding the integrating factor,

\[ \mu(t) = e^{\int (2) \, dt} = e^{2t} \]

Multiplying the differential equation by the integrating factor,

\[ e^{2t}(y' + 2y) = 3e^{3t} \]
\[ \frac{d}{dt}(ye^{2t}) = 3e^{3t} \]

Integrating both sides,

\[ ye^{2t} = e^{3t} + C \]
\[ y(t) = e^t + Ce^{-2t} \]

2.2.8: Find general solutions for the equation

\[ \frac{dy}{dt} + \frac{1}{t}y = \frac{1}{t^2} \]

Solution: Finding the integrating factor,

\[ \mu(t) = e^{\int \frac{1}{t} \, dt} = e^{\ln(t)} = t \]

Multiplying the differential equation by the integrating factor,

\[ ty' + y = \frac{1}{t} \]
\[ \frac{d}{dt}(ty) = \frac{1}{t} \]

Integrating both sides,

\[ ty = \ln(t) + C \]
\[ y(t) = \frac{\ln(t) + C}{t} \]

2.2.14: Find general solutions for the equation

\[ (t^2 + 9) \frac{dy}{dt} + ty = 0 \]

Solution: First, have to divide through by \( y' \) coefficient,

\[ \frac{dy}{dt} + y \frac{t}{(t^2 + 9)} = 0 \]
Finding the integrating factor,

\[ \mu(t) = e^{\int t \left( t^2 + 9 \right)^{-1/2} dt} = e^{1/2 \ln(t^2 + 9)} = (t^2 + 9)^{1/2} \]

Multiplying the differential equation by the integrating factor,

\[ y'(t^2 + 9)^{1/2} + yt(t^2 + 9)^{-1/2} = 0 \]

\[ \frac{d}{dt} \left( y(t^2 + 9)^{1/2} \right) = 0 \]

Integrating both sides,

\[ y(t^2 + 9)^{1/2} = C \]

\[ y(t) = C(t^2 + 9)^{-1/2} \]

2.2.20: Find the solution of the following IVP,

\[ (1 + e^t) \frac{dy}{dt} + e^ty = 0, \quad y(0) = 1 \]

**Solution**: Dividing through by the \( y' \) coefficient,

\[ y' + y \frac{e^t}{1 + e^t} = 0 \]

Finding the integrating factor,

\[ \mu(t) = e^{\int \frac{e^t}{1 + e^t} dt} = e^{\ln(1 + e^t)} = e^t + 1 \]

Multiplying the differential equation by the integrating factor,

\[ (e^t + 1)y' + ye^t = 0 \]

\[ \frac{d}{dt} \left( y(e^t + 1) \right) = 0 \]

Integrating both sides,

\[ y(e^t + 1) = C \]

\[ y(t) = \frac{C}{e^t + 1} \]

Using the initial value,

\[ 1(e^0 + 1) = C \]

\[ C = 2 \]

Hence, the particular solution is

\[ y(t) = \frac{2}{e^t + 1} \]

2.2.28: Solve the differential equation by the integrating factor method, steps 1 – 4

\[ y' + 3t^2y = t^2 \]
Solution:

**Step 1:** Find the integrating factor,

\[ \mu(t) = e^{\int 3t^2 \, dt} = e^{t^3} \]

**Step 2:** Multiply the differential equation by the integrating factor,

\[ e^{t^3} y' + e^{t^3}3t^2 y = t^2 e^{t^3} \]
\[ \frac{d}{dt}(ye^{t^3}) = t^2 e^{t^3} \]

**Step 3:** Find the antiderivative,

\[ ye^{t^3} = \int t^2 e^{t^3} \, dt = \frac{1}{3} e^{t^3} \]

**Step 4:** Solve for \( y \),

\[ y(t) = \frac{1}{3} \]

2.2.29: Solve the differential equation by the integrating factor method, steps 1 – 4

\[ y' + \frac{1}{t} y = \frac{1}{t^2} \]

Solution:

**Step 1:** Finding the integrating factor,

\[ \mu(t) = e^{\int 1/t \, dt} = e^{\ln(t)} = t \]

**Step 2:** Multiplying the differential equation by the integrating factor,

\[ ty' + y = \frac{1}{t} \]
\[ \frac{d}{dt}(ty) = \frac{1}{t} \]

**Step 3:** Integrating both sides,

\[ ty = \ln(t) + C \]

**Step 4:** Solving for \( y \),

\[ y(t) = \frac{\ln(t) + C}{t} \]

2.3.2: **Doubling Time** The time \( t_d \) required for the solution \( y \) of the growth problem

\[ y' = ky, \quad k > 0, \quad y(0) = y_0 \]

to reach twice its original value is called the *doubling time*. Find \( t_d \) in terms of \( k \).
Solution: Solving the differential equation,

$$\int \frac{dy}{y} = \int k \, dt$$

$$\ln(y) = kt + C$$

$$y(t) = Ce^{kt}$$

Using the initial value,

$$y_0 = Ce^0$$

$$y(t) = y_0e^{kt}$$

We want to find the solution that is twice the original value, or $2y_0$. Solving for $t_d$ using this information,

$$2y_0 = y_0e^{kt_d}$$

$$kt_d = \ln(2)$$

$$t_d = \frac{\ln(2)}{k}$$

2.3.5: Determining Decay from Half-Life A certain radioactive substance has a half-life of 5 hours. Find the time for a given amount to decay to one-tenth of its original mass.

Solution: We know the solution to the decay equation,

$$y(t) = y_0e^{kt}$$

Given that the half-life is 5 hours, then we can solve for $y_0$,

$$y(5) = y_0e^{5k} = \frac{1}{2}y_0$$

$$k = -\frac{\ln(2)}{5}$$

Finding $t$ that it takes for the amount to decay to $\frac{1}{10}$ of its original mass,

$$\frac{y_0}{10} = y_0e^{-t \ln(2)/5}$$

$$-t \cdot \ln(2)/5 = -\ln(10)$$

$$t = \frac{5 \cdot \ln(10)}{\ln(2)} \approx 16.6 \text{ hours}$$

2.3.15: Sodium Pentathol Elimination Ed is undergoing surgery for an old football injury and must be anesthetized. The anesthesiologist knows Ed will be “under” when the concentration of sodium pentathol in his blood is at least 50 milligrams per kilogram of body weight. Suppose that Ed weighs 100 kg (220 pounds) and that sodium pentathol is eliminated from the bloodstream at a rate proportional to the amount present. If the half-life of the drug is 10 hours, what single dose should be given to keep Ed anesthetized for three hours?

Solution: From the solution to the decay equation,

$$y(t) = y_0e^{kt}$$

then solving for $k$,

$$\frac{y_0}{2} = y_0e^{k \cdot 10}$$

$$k = -\frac{\ln(2)}{10}$$
Since Ed weighs 100 kg, then we need
\[ 100 \cdot 50 = 5000 \text{ milligrams of sodium pentathol} \]
for Ed to remain asleep by three hours. In other words,
\[ 5000 = y_0 e^{-3 \ln(2)/10} \]
\[ y_0 = 5000 e^{3 \ln(2)/10} \approx 6155.72 \text{ milligrams} \]
Or about 6.16 grams initially.

2.3.29: **How to Become a Millionaire** Upon graduation from college, Sergei has no money. However, during each year after that, he will deposit \( d = \$1,000 \) into an account that pays interest at a rate of 1% compounded continuously.

(a) Find the future value \( A(t) \) of Sergei’s account.

**Solution:** The accumulated value is expressed by the following differential equation
\[ \frac{dA}{dt} = 0.01A + 1000, \quad A(0) = 0 \]
Solving this yields
\[ A(t) = \frac{1000}{0.01} \left( e^{0.01t} - 1 \right) \]

(b) Find the value for an annual deposit \( d \) that would produce a balance of one million dollars when he retires 40 years later.

**Solution:** Because our initial deposit is 0, we want to solve the following equation,
\[ 1000000 = \frac{a}{0.01} \left( e^{0.01 \cdot 40} - 1 \right) \]
for \( a \). Therefore,
\[ a = \frac{10000}{e^{0.01 \cdot 40} - 1} \approx \$4918.25 \]

(c) If \( d = \$2,500 \), what should be the value of the interest rate \( r \) in order for Sergei’s balance to be one million dollars after 40 years?

**Solution:** The previous equation changes to
\[ 1000000 = \frac{2500}{r} \left( e^{r \cdot 40} - 1 \right) \]
Then solving for \( r \), using software to achieve an approximation, we see that \( r \approx 9.04\% \).

2.4.4: **Salty Goal** At the start, 5 lb of salt are dissolved in 20 gal of water. Salt solution with concentration 2 lb/gal is added at a rate of 3 gal/min, and the well-stirred mixture is drained out at the same rate of flow. How long should this process continue to raise the amount of salt in the tank to 25 lb?

**Solution:** We have the following relationship,
\[ \frac{dx}{dt} = \text{RATE IN} - \text{RATE OUT} \]
\[ = (2 \text{ lb/gal}) \cdot (3 \text{ gal/min}) - \left( \frac{x}{20} \text{ lb/gal} \right) \cdot (3 \text{ gal/min}) \]
\[ = 6 - \frac{3}{20} x \]
Solving this differential equation, we can use the method of integrating factor,

\[ \mu = e^{1/20 \int \frac{3}{20} dt} = e^{3t/20} \]

Multiplying this through the original equation, we obtain

\[ \frac{d}{dt} \left( xe^{3t/20} \right) = 6e^{3t/20} \]
\[ xe^{3t/20} = 40e^{3t/20} + C \]
\[ x(t) = 40 + Ce^{-3t/20} \]

Using the initial value, \( x(0) = 5 \),

\[ 5 = 40 + C \quad \Rightarrow \quad C = -35 \]

So

\[ x(t) = 40 - 35e^{-3t/20} \]

Since we want to find the time it takes until there are 25 lbs of salt in the tank, then we have to solve the following equation

\[ 25 = 40 - 35e^{-3t/20} \]
\[ t = \frac{20}{3} \ln \left( \frac{3}{7} \right) \approx 5.65 \text{ mins} \]

2.4.6: **Salty Overflow** A 600-gallon tank is filled with 300 gallons of pure water. A spigot is opened and a salt solution containing 1 lb of salt per gallon of solution begins flowing into the tank at a rate of 3 gal/min. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 1 gal/min. What will be the salt content in the tank at the precise moment that the volume of solution in the tank reaches the tank’s capacity of 600 gal?

**Solution:** We have the following relationship,

\[ \frac{dx}{dt} = \text{RATE IN} - \text{RATE OUT} \]
\[ = (1 \text{ lb/gal}) \cdot (3 \text{ gal/min}) - \left( \frac{x}{300 + (3 - 1)t} \text{ lb/gal} \right) \cdot (1 \text{ gal/min}) \]
\[ = 3 - \frac{x}{300 + 2t} \]

Using the method of integrating factors,

\[ \mu(t) = e^{1/2 \ln(300 + 2t)} dt = e^{1/2 \ln(300 + 2t)} = (300 + 2t)^{1/2} \]

Multiplying the original differential equation,

\[ (300 + 2t)^{1/2} x' + (300 + 2t)^{-1/2} x = 3(300 + 2t)^{1/2} \]
\[ \frac{d}{dt} \left( (300 + 2t)^{1/2} x \right) = 3(300 + 2t)^{1/2} \]
\[ (300 + 2t)^{1/2} x = (300 + 2t)^{3/2} + C \]
\[ x(t) = (300 + 2t) + C(300 + 2t)^{-1/2} \]

From the initial value, we know that \( x(0) = 0 \), so

\[ 0 = (300 + 2 \cdot 0) + C(300 + 2 \cdot 0)^{-1/2} \]
\[ C(300)^{-1/2} = (300) \]
\[ C = -300^{3/2} \]
We know that 2 gal/min are being added, and we need to fill 300 gals to meet the capacity, so it will take 150 minutes to fill the tank. Therefore,

\[ x(150) = (300 + 2 \cdot 150) - 300^{3/2} \cdot (300 + 2 \cdot 150)^{-1/2} \approx 387.868 \text{ lbs} \]

**2.4.15: Using the Time Constant** At noon, with the temperature in your house at 75°F and the outside temperature at 95°F, your air conditioner breaks down. Suppose that the time constant 1/k for your house is 4 hours.

(a) What will the temperature in your house be at 2:00 pm?

**Solution:** Our differential equation becomes

\[ \frac{dT}{dt} = \frac{1}{4} (95 - T) \]

Solving,

\[
\int \frac{dT}{95 - T} = \int \frac{dt}{4} \\
- \ln|95 - T| = \frac{t}{4} + C \\
T(t) = 95 - Ce^{-t/4}
\]

Using the initial condition, \( T(0) = 75 \), then

\[ 75 = 95 - C \implies C = 20 \]

Then, at \( t = 2 \),

\[ T(2) = 95 - 20e^{-2/4} \approx 82.97 \text{ book’s solution is off, see second part} \]

(b) When will the temperature in your house reach 80°F?

**Solution:** We need to solve the equation,

\[ 80 = 95 - 20e^{-t/4} \]

for \( t \). Therefore,

\[ e^{-t/4} = \frac{15}{20} \]

\[ t = -4\ln\left(\frac{3}{4}\right) \approx 1.15 \text{ hours} \]

Consider 0.15 \cdot 60 \approx 9 minutes, so we expect it to be 80°F at about 1:09 pm.

**2.4.19: The Coffee and Cream Problem** John and Maria are having dinner, and each orders a cup of coffee. John cools his coffee with some cream. They wait 10 minutes and then Maria cools her coffee with the same amount of cream. The two then began to drink. Who drinks the hotter coffee?

**Solution:** Define \( t_J \) as the time that John pours cream into his coffee, and let \( t_M \) be the time Maria pours her cream, so \( t_M = t_J + 10 \).

At \( t_J \), we know that John’s coffee is 10°F cooler than Maria’s coffee. In between \( t_J \) and \( t_M \) (in the ten minute interval), John’s coffee is cooling more slowly than Maria’s coffee, and Maria’s coffee is always hotter than John’s. At \( t_M \), we suspect that there will be less than a 10°F difference between John and Maria’s coffee temperatures. So when Maria adds cream to her coffee, it will cause her coffee temperature to be lower than John’s.

Hence, John drinks the warmer coffee.