ACM 11: Homework 5

Assigned Monday, Nov 3 2008. Due on Wednesday, Nov 12 2008 at noon. 50 pts.

Submission instructions: follow the format of the Mathematica problem set template in the handout section, and submit the notebook file in the format `Firstname_Lastname_1.nb` to `ftp.its.caltech.edu/pub/srbecker/incoming`. Follow the standard instructions for resubmissions.

1. **Using Mathematica as a calculator** (10 pts) You can do the following problems however you like. It may help to look at the Wikipedia entry on Fibonacci numbers. Show your work for whichever method you choose.

   (a) Is 190392490709135 a Fibonacci number?

   (b) What about 1500520536206896083287?

   (c) Is 82228386541779228177255628800000000 a factorial?

   (d) What about 3345252661316380710817006205344075166515200000000000?

   (e) Calculate $\sum_{j=1}^{n} j\cos(j)$.

2. **Surface area of n-spheres** (20 pts) The expression for the surface area of the n-sphere is

$$A = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

   Recall that if n is an integer, $\Gamma(n+1) = n!$; in Mathematica, the Γ function is represented by `Gamma`. Using Stirling's formula, we find the approximation $\Gamma\left(\frac{n}{2}+1\right) \sim \sqrt{2\pi}e^{-n/2}\left(\frac{n}{2}\right)^{(n+1)/2}$. Call A' the expression obtained when the denominator is replaced with this approximation.

   We are interested in the behavior of the surface area as a function of n.

   (a) Use `Table` and `ListPlot` to see how A and A' behave between $n = 1$ and $n = 20$. Color A blue and A' purple. Which is larger?

   (b) Just for kicks, plot A and A' not as functions of discrete n, but as functions of continuous x on the same range. Using the help, determine how to use `Filling` to fill the area between A and A' and do so.

   (c) Use `Limit` to verify that as $n \to \infty$, $A' \to A$ in the sense that their ratio goes to unity.

   (d) Use `D` to calculate an expression for the rate of change of the surface area A as a function of x.

   (e) Attempt to find the real number x for which A is maximum by solving $\frac{dA}{dx} = 0$. You have multiple options here: `Solve`, `Reduce`, `NSolve`, and `FindRoot` are all commands which might work in this context. Attempt them all, and consider what the results tell you about the limitations of Mathematica .

   (f) Plot $\frac{dA}{dx}$ over the same range of n (discrete values) as the first plot. Considering A as a function over discrete n, what is the maximum value of A, and at what value of n does it occur?

3. **Monte Carlo computations** (20 pts)

   (a) If we randomly choose points out of the square $[-1,1]\times[-1,1]$, geometric reasoning tells us that the probability of choosing points in the unit circle $B(0,1)$ is $\pi/4$:

$$\mathbb{P}(\text{random point lies in the unit circle}) = \mathbb{E}\chi_{\{(x,y):x^2+y^2\leq 1\}}(x,y) = \pi/4$$

   where $\chi_{\{(x,y):x^2+y^2\leq 1\}}$ is the indicator function for the unit circle (so has the value 1 when (x,y) is in the circle and 0 otherwise), and the expectation is taken over the points uniformly distributed in $[-1,1]\times[-1,1]$. Let's use this to crudely approximate π.

The syntax of the conditional expression in Mathematica is `If[expr, true_expr, false_expr]`: if `expr` evaluates to true, then the value of the `If` expression is that of `true_expr`, otherwise it is that of `false_expr`. So for example, the value of `If[x <= 0, -x, x]` is $|x|$.

Use `Table`, `RandomReal`, and `If` to generate a list of 10^5 random samples of $\chi_{\{(x,y):x^2+y^2\leq 1\}}$ over the square. Use `Mean` to take the average of these, and use this to calculate an approximate value of π. You can do this entire calculation on one line.

(b) It turns out the probability of two positive random integers being coprime is $6/\pi^2$. Using the same technique as above to approximate this probability, estimate the value of π; you may find the `CoprimeQ` or `GCD` functions useful. Again use 10^5 samples; to sample the positive random numbers, choose numbers randomly in the range $[2, 10^5]$.