ACM 11: Homework 7

Assigned Tuesday, Nov 18 2008. Due on Wednesday, Nov 26 2008 at noon. 50 pts.

Submission instructions: follow the format of the Mathematica problem set template in the handout section, and submit the notebook file in the format `Firstname_Lastname_3.nb` to `ftp.its.caltech.edu/pub/srbecker/incoming`. Follow the standard instructions for resubmissions.

1. **Function definitions** (20 points)

   (a) Define a function `colnorm` which takes a matrix `A` as its argument and calculates the column norm of `A`: the sum of the Euclidean norms of the columns of `A`. Recall that the Euclidean norm of a vector $a \in \mathbb{R}^n$ is $\|a\|_2 = \sqrt{\sum_{i=1}^n a_i^2}$, and can be computed with the `Norm` function.

   Determine how to generate Hilbert matrices in Mathematica , and determine the column norm of the 7×7 Hilbert matrix. (You can use the result to check your function definition).

   (b) Define a function `onetwonorm` which takes a matrix `A` as its argument and calculates $\|A\|_{1\to 2} = \max_j \|A_j\|_2$, the largest of the Euclidean norms of the columns of `A`. You may find the `Ordering`, `Sort`, or `Max` functions useful (see the help for details on how to use these functions).

   Determine how to generate Toeplitz matrices in Mathematica , and determine the one-two norm of the Toeplitz matrix whose first column is $(1, -3, 4, 6, 2)^t$.

   (c) Define a function ∞`twonorm` which takes a matrix `A` as its argument and calculates $\|A\|_{\infty\to 2} = \max_{v\in B_\infty} \|Av\|_2$ where B_∞ is the set of vectors, of appropriate length, whose components are all 1 or -1.

   Use the command `Tuple` for the hard part: generating a list of the vectors in B_∞.

   The most natural solution to this problem involves anonymous functions. Analogous definitions which use this route are given in the notes for the ∞-1 norm– you may refer to these while writing ∞`twonorm`. If you do, include a *detailed* explanation showing you understand why your code works.

   On the other hand, it is very much possible to do this without using anonymous functions. However, your function will probably involve more than one line of code, and at least one auxiliary variable, so use `Module` or `Block`.

   Determine the ∞-2 norm of the 6×6 Fourier matrix F whose entries are given by $F_{jk} = \frac{1}{\sqrt{6}} e^{\pi i jk/3}$. Recall that $i = \sqrt{-1}$ is represented as `I` or ESC-ii-ESC in Mathematica

2. **Functions and Patterns** (30 points with a 10 point bonus) Let $P_n(z) = \sum_{k=0}^n a_k z^k$ be a polynomial of degree n. We'd like to investigate the behavior of the integral

$$\frac{1}{2\pi} \int_0^{2\pi} F_n(\theta - \varphi) e^{in(\theta-\varphi)} P_n(e^{i\varphi})\, d\varphi$$

   where

$$F_n(\theta) = \sum_{j=-n+1}^{n-1} (n - |j|) e^{ij\theta}.$$

   Again, remember that $i = \sqrt{-1}$ is entered in Mathematica as `I` or with the shortcut ESC-ii-ESC.

   (a) Use Mathematica to find a reduced expression for $F_n(\theta)$. Note that if you enter it directly and use `FullSimplify`, the result is not very appealing. Instead, let's do a few auxiliary manipulations: convince yourself that

$$F_n(\theta) = n + 2 \sum_{j=1}^{n-1} (n - j) \cos(j\theta)$$

then use `TrigToExp` and `FullSimplify` to find a very simple expression for F_n. Your result should be a rational function in terms of $\cos(n\theta)$ and $\cos(\theta)$.

(An explanation: `TrigToExp` converts expressions in terms of trigonometric functions to ones in terms of complex exponentials (incidentally, `ExpToTrig` does the opposite). These can be simplified with simple algebra, whereas to simplify trig expressions, Mathematica must apply a whole set of rules– the end result is you have a much better chance of getting good simplifications if you first convert trig expressions to exponential expressions before attempting simplification. Note also that `FullSimplify` will automatically convert complex exponentials into trigonometric functions whenever possible.) The point to take home from this is that often times, when computing nontrivial simplifications, you need to help Mathematica out.

(b) Create a function `F` that takes θ and `n` as arguments and computes the expression you found above for F_n.

(c) Write a function `polyint` that takes a function `p` and an angle θ as its arguments and computes the integral

$$\frac{1}{2\pi}\int_0^{2\pi} F_n(\theta-\varphi)e^{in(\theta-\varphi)}P_n(e^{i\varphi})\,d\varphi,$$

assuming that `p` defines a polynomial. Hint: you can use the function `CoefficientList`, which returns a list of the coefficients of a polynomial, to determine the degree of `p`; you might want to use a `Module` and store this degree in a local variable `n`.

(d) Define the function `p1` which accepts a single argument x and computes the polynomial ax^3+bx^2+cx+d.

(e) Compute `polyint[p1, `θ`]`. Note that this is a polynomial in $e^{i\theta}$. Mentally convert it to a polynomial in $x = e^{i\theta}$; from this, tell us what you think the integral formula does to an arbitrary polynomial.

(f) **Bonus.** Use a pattern to convert the result of `polyint[p1, `θ`]` into a polynomial in `x`.