Trig identities

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11/15/05

Definitions:

$$\begin{aligned}\tan(x) &= \sin(x)/\cos(x)\\ \csc(x) &= 1/\sin(x)\\ \sec(x) &= 1/\cos(x)\\ \cot(x) &= 1/\tan(x)\end{aligned}$$

Pythagorean:

$$\begin{aligned}\sin^2(x)+\cos^2(x) &= 1\\ 1+\cot^2(x) &= \csc^2(x)\\ 1+\tan^2(x) &= \sec^2(x)\end{aligned}$$

Cofunction:

$$\begin{aligned}\sin(\pi/2-x) &= \cos(x) & \cos(\pi/2-x) &= \sin(x)\\ \tan(\pi/2-x) &= \cot(x) & \cot(\pi/2-x) &= \tan(x)\\ \sec(\pi/2-x) &= \csc(x) & \csc(\pi/2-x) &= \sec(x)\end{aligned}$$

Difference/sum:

$$\begin{aligned}\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)\\ \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)\\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\\ \sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ \tan(\alpha-\beta) &= \frac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}\\ \tan(\alpha+\beta) &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\end{aligned}$$

Double angle:

$$\begin{aligned}\sin(2x) &= 2\sin(x)\cos(x)\\ \tan(2x) &= \frac{2\tan(x)}{1-\tan^2(x)}\\ \cos(2x) &= \cos^2(x)-\sin^2(x)\\ &= 2\cos^2(x)-1\\ &= 1-2\sin^2(x)\end{aligned}$$

*last updated Jan 30 2014

Half angle:

$$\begin{aligned}\sin(x/2) &= \pm\sqrt{1/2(1-\cos(x))}\\ \cos(x/2) &= \pm\sqrt{1/2(1+\cos(x))}\\ \tan(x/2) &= \frac{1-\cos(x)}{\sin(x)}\\ &= \frac{\sin(x)}{1+\cos(x)}\end{aligned}$$

Product/sum:

$$\begin{aligned}2\cos(\alpha)\cos(\beta) &= \cos(\alpha-\beta)+\cos(\alpha+\beta)\\ 2\sin(\alpha)\sin(\beta) &= \cos(\alpha-\beta)-\cos(\alpha+\beta)\\ 2\sin(\alpha)\cos(\beta) &= \sin(\alpha+\beta)+\sin(\alpha-\beta)\\ 2\cos(\alpha)\sin(\beta) &= \sin(\alpha+\beta)-\sin(\alpha-b)\end{aligned}$$

Sum/product:

$$\begin{aligned}\sin(2\alpha)+\sin(2\beta) &= 2\sin(\alpha+\beta)\cos(\alpha-\beta)\\ \sin(2\alpha)-\sin(2\beta) &= 2\cos(\alpha+\beta)\sin(\alpha-\beta)\\ \cos(2\alpha)+\cos(2\beta) &= 2\cos(\alpha+\beta)\cos(\alpha-\beta)\\ \cos(2\alpha)-\cos(2\beta) &= -2\sin(\alpha+\beta)\sin(\alpha-\beta)\end{aligned}$$

Power reducing:

$$\begin{aligned}\sin^2(x) &= (1-\cos(2x))/2\\ \cos^2(x) &= (1+\cos(2x))/2\\ \tan^2(x) &= \frac{1-\cos(2x)}{1+\cos(2x)}\end{aligned}$$

Hyperbolic:

$$\begin{aligned}\sinh(iz) = i\sin(z), &\quad \sin(iz) = i\sinh(z) \quad (z\in\mathbb{R})\\ \cosh(iz) = \cos(z), &\quad \cos(iz) = \cosh(z) \quad (z\in\mathbb{R})\\ \cosh^2(z)-\sinh^2(z) &= 1 \quad (z\in\mathbb{C})\end{aligned}$$

Exponential formulae:

$$\begin{aligned}\sin(z) &= (e^{iz}-e^{-iz})/(2i)\\ \cos(z) &= (e^{iz}+e^{-iz})/2\\ \sinh(z) &= (e^{z}-e^{-z})/2\\ \cosh(z) &= (e^{z}+e^{-z})/2\end{aligned}$$

Derivatives:

$$\frac{d}{dx}\sin(x) = \cos(x)dx$$
$$\frac{d}{dx}\cos(x) = -\sin(x)dx$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)dx$$
$$\frac{d}{dx}\cot(x) = -\csc^2(x)dx$$
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)dx$$
$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)dx$$
$$\frac{d}{dx}\sin^2(x) = 2\sin(x)\cos(x)dx$$
$$\frac{d}{dx}\cos^2(x) = -2\sin(x)\cos(x)dx$$
$$\frac{d}{dx}\tan^2(x) = 2\tan(x)\sec^2(x)dx$$
$$\frac{d}{dx}\arccos(x) = -(1-x^2)^{-\frac{1}{2}}dx$$
$$\frac{d}{dx}\arcsin(x) = (1-x^2)^{-\frac{1}{2}}dx$$
$$\frac{d}{dx}\arctan(x) = (1+x^2)^{-1}dx$$
$$\frac{d}{dx}\cosh(x) = \sinh(x)dx$$
$$\frac{d}{dx}\sinh(x) = \cosh(x)dx$$

Mclaurin Series (and Laurent Series about origin):

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}$$
$$= x - x^3/3! + x^5/5! - ...$$
$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$
$$= 1 - x^2/2! + x^4/4! - ...$$
$$\tan(x) = x + 1/2\cdot x^3 + 2/15\cdot x^5 + 17/315\cdot x^7 + ...$$
$$= \sum_{n=1}^{\infty}\frac{B_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1} \quad \text{for} \quad |x| < \frac{\pi}{2}$$

[B_n is the n^{th} Bernoulli number]

$$\cot(z) = z^{-1} - z/3 - z^3/45 - 2/945\cdot z^5 - ...$$

[for $0 < |z| < \pi$]

$$\csc(z) = z^{-1} + z/6 + 7/360\cdot z^3 + .002\cdot z^5 + ... \text{ for } 0 < |z| < \pi$$
$$\sec(z) = 1 + z^2/2 + 5/24\cdot z^4 + ... \text{ for } |z| < \pi/2$$

Misc. Functions

Sine Cardinal:

$$\operatorname{sinc}(x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ \frac{\sin(x)}{x} & \text{otherwise} \end{cases}$$

Sine Integral:

$$\operatorname{Si}(z) = -\int_0^z \frac{\sin(t)}{t}dt$$

Integrals

$$\int_0^x \cos^2(\theta)d\theta = 1/2x + 1/4\sin(2x)$$
$$\int_0^x \sin^2(\theta)d\theta = 1/2x - 1/4\sin(2x)$$
$$\int_0^\pi \sin^2(x)dx = \pi/2$$
$$= \int_0^\pi \cos^2(x)dx$$
$$\int_0^{2\pi} \sin^2(x)dx = \pi$$
$$= \int_0^{2\pi} \cos^2(x)dx$$

Inverse Functions

$$\cos^{-1}(z) = -i\log(z + i(1-z^2)^{1/2})$$
$$\sin^{-1}(z) = -i\log(iz + (1-z^2)^{1/2})$$
$$\tan^{-1}(z) = \frac{1}{2i}\log\frac{i-z}{i+z}$$
$$\cosh^{-1}(z) = \log(z + (z^2-1)^{1/2})$$
$$\sinh^{-1}(z) = \log(z + (1+z^2)^{1/2})$$
$$\tanh^{-1}(z) = \frac{1}{2}\log\frac{1+z}{1-z}$$

Binomial Theorem

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\begin{pmatrix}\alpha \\ n\end{pmatrix}x^n \text{ for } |x| < 1,\ \alpha \in \mathbb{C}$$

Unit circle (taken from `http://www.texample.net/tikz/examples/unit-circle/` by Supreme Aryal, under creative commons license 2.5)