The Inversion Problem and Applications of the Generalized Radon Transform

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Abstract

We prove that under certain conditions the inversion problem for the generalized Radon transform reduces to solving a Fredholm integral equation and we obtain the asymptotic expansion of the symbol of the integral operator in this equation.

We consider applications of the generalized Radon transform to partial differential equations with variable coefficients and provide a solution to the inversion problem for the attenuated and exponential Radon transforms.

Introduction

The classical Radon transform of a function u is the function $R_c u$ defined on a family of hyperplanes; the value of the $R_c u$ on a given hyperplane is the integral of u over that hyperplane.

Such a transform was studied first by J. Radon [15]. The Radon transform and its applications were studied by I. M. Gel'fand [2], [3], F. John [8], S. Helgason [6], [7], D. Ludwig [12], P. D. Lax and R. S. Phillips [10], [11] and others (for more references see [7] for example). I. M. Gel'fand, M. I. Graev and N. Ya. Shapiro [3] introduced a general topological framework (the notion of double fibration) for generalized Radon transform R and its dual R^*. V. Guillemin [5] has shown that R^*R is an elliptic pseudodifferential operator. E. Quinto [14] studied the dependence of the operator R^*R on the defining measures.

In this paper we consider the inversion problem for the generalized Radon transform for the special case of double fibration when instead of the family of hyperplanes (as in the classical case) one has a family of hypersurfaces. We prove that the inversion problem can be reduced to solving a Fredholm integral equation.

In the first section we define the generalized Radon transform R and its dual R^*. Our definition is a natural generalization of the classical one. We consider hypersurfaces instead of hyperplanes and introduce densities into the transforms.

To solve the inversion problem we analyze the Fourier integral operator F with a special choice of phase function and amplitude and we prove that the operator F can be factored as $F = R^*KR$, where K is a one-dimensional operator of convolution type. We show that by choosing R^* and K properly the operator F can be made equal to $I + T$, where T is a compact operator in an appropriate space. We obtain the asymptotic expansion of the symbol of the operator T. Also we show that if the generalized Radon transform is a small perturbation of the classical one, then the operator T is small. (In the case of the classical Radon transform $T = 0$.)

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In Section 6 we define the transform R_λ which is a further generalization of the Radon transform.

In the following section we consider applications of the transform R_λ to partial differential equations. Applications of the classical Radon transform to partial differential equations can be found, for example, in [1], [2], [6], [10], [11]. The basis for such applications is the fact that the composition of the classical Radon transform and a partial differential operator with constant coefficients transforms the latter into an ordinary differential operator.

In particular, the solution of the Cauchy problem for the wave equation can be represented with the help of the classical Radon transform (see [10], for example). Having this in mind, we prove that the composition of the generalized Radon transform with a partial differential operator with variable coefficients transforms the latter (locally) into an ordinary differential operator up to some smooth discrepancy.

Also, we demonstrate in the last section how the inversion problem for attenuated and exponential Radon transforms can be solved using the approach developed in this paper.

1. Generalized Radon Transform and Its Dual

In this section we give definitions of the generalized Radon transform and its dual.

To describe a surface in n-dimensional Euclidean space $\mathbb{R}^n$ we introduce a function $\phi(x, \theta)$ defined on $X \times (\mathbb{R}^n \backslash \{0\})$, where X is a domain in $\mathbb{R}^n$. We assume that the function $\phi(x, \theta)$ satisfies the following conditions:

(i) $\phi(x, \theta)$ is a real-valued C^∞ function on $X \times (\mathbb{R}^n \backslash \{0\})$.

(ii) $\phi(x, \theta)$ is homogeneous with respect to θ of degree one: $\phi(x, \lambda\theta) = \lambda\phi(x, \theta)$ for real λ.

(iii) $d_x\phi$, the differential of ϕ with respect to x, does not vanish anywhere in $X \times (\mathbb{R}^n \backslash \{0\})$.

(iv) The function $h(x, \theta) > 0$ in $X \times (\mathbb{R}^n \backslash \{0\})$, where

$$h(x, \theta) = \det\left(\frac{\partial^2\phi(x, \theta)}{\partial x^j\, \partial\theta^k}\right).$$

In the case of the classical Radon transform $\phi(x, \theta) = x \cdot \theta$, where "$\cdot$" denotes the standard inner product in $\mathbb{R}^n$.

Given ϕ, we construct a family $\mathscr{H}$ of $(n-1)$-dimensional surfaces $H_{s,\omega}$ in the domain X, where

$$H_{s,\omega} = \{x \in X \mid s = \phi(x, \omega),\ s \in R^1,\ \omega \in S^{n-1}\}.$$

As follows from (ii), $H_{s,\omega} = H_{-s,-\omega}$. Hence, $\mathscr{H}$ can be identified with $(\mathbb{R}^1 \times S^{n-1})/Z_2$.

We denote by $[s, \omega]$ elements of $\mathscr{H} = (\mathbb{R}^1 \times S^{n-1})/Z_2$. For given $x \in X$ let Y_x be the set of all hypersurfaces from the family $\mathscr{H}$ passing through the point $x \in X$,

$$Y_x = \{[\phi(x, \omega), \omega] \in \mathscr{H} \mid \omega \in S^{n-1}\}.$$

In order to have double fibration (see [3]), two conditions must be satisfied:

(v) if $H_{s,\omega} = H_{s',\omega'}$, then $[s, \omega] = [s', \omega']$.

(vi) if $Y_x = Y_{x'}$, then $x = x'$.

In our case, (v) and (vi) are consequences of the conditions (iii), (iv) and the implicit function theorem.

Now we can introduce the generalized Radon transform R and its dual R^*. For functions $u \in C_0^\infty(X)$ we define the generalized Radon transform R as follows:

$$(1.1) \qquad (Ru)([s, \omega]) = \int_{H_{s,\omega}} u(x) a(x, \omega) \Omega,$$

where Ω is the differential form

$$(1.2) \qquad \Omega = \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial \phi(x, \omega)/\partial x^j}{|\nabla_x \phi(x, \omega)|^2} dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n.$$

The differential form Ω has been chosen to satisfy the following identity

$$(1.3) \qquad d_x \phi(x, \omega) \wedge \Omega = dx^1 \wedge \cdots \wedge dx^n.$$

The density $a(x, \omega)$ in (1.1) is a positive function on $X \times S^{n-1}$. We assume that the density $a(x, \omega)$ belongs to $C^\infty(X \times S^{n-1})$ and satisfies the condition $a(x, \omega) = a(x, -\omega)$.

To introduce the dual transform let us consider the space of functions

$$C^\infty(\mathscr{H}) = \{v(s, \omega) \in C^\infty(\mathbb{R}^1 \times S^{n-1}) \mid v(s, \omega) = v(-s, -\omega)\}.$$

For functions $v \in C^\infty(\mathscr{H})$ we define the dual transform R^* as follows:

$$(1.4) \qquad (R^* v)(y) = \int_{|\omega|=1} b(y, \omega) v(s, \omega)|_{s=\phi(y,\omega)} \, d\omega,$$

where the density $b(y, \omega)$ is a positive function on $X \times S^{n-1}$. We also assume that the density $b(y, \omega)$ belongs to $C^\infty(X \times S^{n-1})$ and satisfies the condition $b(y, \omega) = b(y, -\omega)$.

Sometimes (in tomography, for example) the dual of the classical Radon transform is called the backprojection operator; hence, by analogy, one can call R^* the generalized backprojection operator.

We shall use the notation $(Ru)(s, \omega)$ for the generalized Radon transform (1.1). We note that the relation

$$(Ru)(-s, -\omega) = (Ru)(s, \omega)$$

is always satisfied.

2. The Fourier Integral Operator F

We denote by $a(x, \theta)$ and $b(y, \theta)$ the extensions of the densities $a(x, \omega)$ and $b(y, \omega)$ on the space $X \times (\mathbb{R}^n \setminus \{0\})$ by the formulae

$$a(x, \theta) = a(x, \omega),$$
$$b(y, \theta) = b(y, \omega),$$

where $\theta \neq 0$ and $\omega = \theta / |\theta|$.

Let $U(s)$ be an infinitely differentiable real function which has the following properties:

$$U(s) = U(-s)$$

and

$$|\partial_s^k U(s)| \leqq C(k) \langle s \rangle^{m-k},$$

where m is an arbitrary real number and $\langle s \rangle = (1 + s^2)^{1/2}$.

We define the amplitude $A(x, y, \theta)$ by the formula

$$A(x, y, \theta) = a(x, \theta) b(y, \theta) U(|\theta|).$$

We can always choose the function U (by setting $U(|\theta|) = 0$ in the neighborhood of the point $\theta = 0$) in such a way that $A(x, y, \theta)$ belongs to $S^m(X \times X \times R^n)$, i.e., $A(x, y, \theta) \in C^\infty(X \times X \times R^n)$ and for every compact $Q \subset X \times X$ and for every three multiindices α, β, γ there is a constant $C_Q(\alpha, \beta, \gamma)$ such that

$$|\partial_\theta^\alpha \partial_x^\beta \partial_y^\gamma A(x, y, \theta)| \leqq C_Q(\alpha, \beta, \gamma) \langle \theta \rangle^{m - |\alpha|},$$

where $\langle \theta \rangle = (1 + |\theta|^2)^{1/2}$.

For functions $u \in C_0^\infty(X)$ we introduce the Fourier integral operator F as follows:

$$(Fu)(y) = \frac{1}{(2\pi)^n} \int_{R^n} \int_X e^{i\Phi(x,y,\theta)} A(x, y, \theta) u(x)\, dx\, d\theta, \tag{2.1}$$

where

$$\Phi(x, y, \theta) = \phi(x, \theta) - \phi(y, \theta). \tag{2.2}$$

The standard procedure can be applied (see [16], [18]) to regularize the integral at the right-hand side of (2.1), if it is necessary.

Let us consider the set C_Φ,

$$C_\Phi = \{(x, y, \theta) \mid \nabla_\theta \Phi(x, y, \theta) = 0, x \in X, y \in X, \theta \in R^n \setminus \{0\}\}.$$

It follows from the definition of $\Phi(x, y, \theta)$ in (2.2) and condition (iv) that in our case

$$C_\Phi = \{(x, x, \theta) \mid x \in X, \theta \in R^n \setminus \{0\}\}.$$

Remark. In further arguments we shall deal with amplitudes $A(x, y, \theta)$ which belong to $C^{\infty}(X \times X \times (\mathbb{R}^n \setminus \{0\}))$. We shall treat them as if they were from the class $C^{\infty}(X \times X \times \mathbb{R}^n)$, since we can always change them in the neighborhood of the point $\theta = 0$ to obtain the desired property, while the operator F will be changed only by a regularizing operator.

Moreover, we can define the Fourier integral operator F in (2.1) for $A(x, y, \theta) \in S^m(X \times X \times (\mathbb{R}^n \setminus \{0\}))$.

3. Factorization of the Fourier Integral Operator F

We introduce spherical coordinates in (2.1), so that $d\theta = r^{n-1}\, dr\, d\omega$ and we can write the Fourier integral operator in (2.1) as

$$(Fu)(y) = \int_{|\omega|=1} G(y, \omega)\, d\omega,$$

where

$$G(y, \omega) = \frac{1}{(2\pi)^n} \int_0^{\infty} \left(\int_X e^{i\Phi(x,y,r\omega)} A(x, y, r\omega) u(x)\, dx \right) r^{n-1}\, dr.$$

The function $G(y, \omega)$ is integrated over the sphere $|\omega| = 1$. Hence, it is sufficient to consider only its even component in ω, i.e.,

$$G^{\text{even}}(y, \omega) = \tfrac{1}{2}(G(y, \omega) + G(y, -\omega)).$$

We have

$$G^{\text{even}}(y, \omega) = \frac{1}{2(2\pi)^n} \int_{-\infty}^{+\infty} \left(\int_X e^{i\Phi(x,y,r\omega)} A(x, y, r\omega) u(x)\, dx \right) |r|^{n-1}\, dr.$$

By virtue of (1.3), $dx = ds \wedge \Omega$, where $s = \phi(x, \omega)$. Now using (1.1) and (2.2) we obtain

$$G^{\text{even}}(y, \omega) = \frac{1}{2(2\pi)^n} \int_{-\infty}^{+\infty} \left(e^{-ir\phi(y,\omega)} b(y, \omega) U(r) \int_{-\infty}^{+\infty} e^{irs} (Ru)(s, \omega)\, ds \right) |r|^{n-1}\, dr.$$

Applying Fubini's theorem, we have

$$(Fu)(y) = \int_{|\omega|=1} d\omega\, b(y, \omega) \left(\int_{-\infty}^{+\infty} (Ru)(s, \omega) K(s - s')|_{s' = \phi(y,\omega)}\, ds \right), \tag{3.1}$$

where

$$K(s) = \frac{1}{2(2\pi)^n} \int_{-\infty}^{+\infty} |r|^{n-1} U(r)\, e^{irs}\, dr. \tag{3.2}$$

We note that if $U(r)$ does not decay sufficiently fast, then $K(s)$ is a generalized function. Let K denote the operator with the generalized kernel $K(s - s')$. We can write (3.1) as

$$F = R^*KR. \tag{3.3}$$

We summarize our result as

THEOREM 1. *The Fourier integral operator F in* (2.1) *can be factored into the form* (3.3), *where R is the generalized Radon transform* (1.1), *R^* its dual* (1.4), *and K is the operator with the kernel* (3.2).

Theorem 1 is a generalization of Theorem 1.1 of [12].

4. Inversion of the Generalized Radon Transform

In this section we prove that the operator F in (2.1) is a pseudo-differential operator. Further, we show that, given functions $\phi(x, \theta)$ and $a(x, \omega)$ in (1.1), we can define the density $b(y, \omega)$ in (1.4) and the function $U(r)$, so that the operator F will be "almost" the identity (up to a less singular operator).

Condition (iii) implies that F maps $C_0^\infty(X)$ continuously into $C^\infty(X)$. It also implies that the map defined by the integral in (2.1) can be extended as a continuous operator

$$F: \mathscr{E}'(X) \to \mathscr{D}'(X),$$

where $\mathscr{D}'(X)$ is the space of distributions on X (the dual of $C_0^\infty(X)$) and $\mathscr{E}'(X)$ is the space of distributions with compact support (the dual of $C^\infty(X)$). We shall say that an operator is regularizing if it maps $\mathscr{E}'(X)$ into $C^\infty(X)$.

Let $\mathscr{L}^m(X)$ be the class of standard pseudo-differential operators of order m. The Fourier integral operator belongs to $\mathscr{L}^m(X)$ if it has phase function $\Phi(x, y, \theta) = (x - y) \cdot \theta$ and amplitude $A(x, y, \theta) \in S^m(X \times X \times \mathbb{R}^n)$.

We denote by $\mathscr{L}^{-\infty}(X)$ the intersection of all $\mathscr{L}^m(X)$, where m is real. Every pseudo-differential operator from the class $L^{-\infty}(X)$ is regularizing and every regularizing operator can be represented as an operator from the class $\mathscr{L}^{-\infty}(X)$ (see [18] for example).

After these preliminary remarks we are able to state the following theorem.

THEOREM 2. (i) *The operator F in* (2.1) *is a pseudo-differential operator. If $A(x, y, \theta) \in S^m(X \times X \times \mathbb{R}^n)$, then $F \in \mathscr{L}^m(X)$.*

(ii) *Set $b(y, \theta) = h(y, \theta)/a(y, \theta)$ and $U(|\theta|) = 1$ in* (2.1), *where $h(y, \theta)$ is given by condition* (iv). *Then the operator defined in* (2.1) *can be extended to the operator*

$$F: L^2(X, \text{compact}) \to L^2(X, \text{loc}),$$

so that

$$F = I + T,$$

where T is a compact operator.

Proof: We can always find a function $\chi_\varepsilon(x,y)\in C^\infty(X\times X)$ such that $0\leqq\chi_\varepsilon\leqq 1$ and

$$\chi_\varepsilon(x,y)=1 \quad \text{if} \quad |x-y|<\tfrac{1}{2}\varepsilon,$$
$$\chi_\varepsilon(x,y)=0 \quad \text{if} \quad |x-y|>\varepsilon,$$

for $\varepsilon>0$. We can write the operator F as a sum $F=F_\varepsilon+\tilde{F}$, where

$$(F_\varepsilon u)(y)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_X e^{i\Phi(x,y,\theta)}A(x,y,\theta)\chi_\varepsilon(x,y)u(x)\,dx\,d\theta,$$

and

$$(\tilde{F}u)(y)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_X e^{i\Phi(x,y,\theta)}A(x,y,\theta)(1-\chi_\varepsilon(x,y))u(x)\,dx\,d\theta.$$

Since the amplitude $A(x,y,\theta)(1-\chi_\varepsilon(x,y))$ vanishes in the conic neighborhood of the set C_Φ, the operator $\tilde{F}$ is regularizing.

To prove that the operator F_ε is pseudo-differential we shall use the following theorem (Theorem 19.1 of [16]):

THEOREM 3. *Let X be a domain in $\mathbb{R}^n$. Let $\psi(x,y,\xi)$ be the phase function defined on $X\times X\times(\mathbb{R}^n\backslash\{0\})$ and $A(x,y,\xi)\in S^m(X\times X\times\mathbb{R}^n)$ be the amplitude of the Fourier integral operator $\hat{F}$.*

If

(a) $\nabla_\xi\psi(x,y,\xi)=0$ *if and only if* $x=y$,

(b) $\nabla_x\psi(x,x,\xi)=\xi$,

then $\hat{F}\in\mathscr{L}^m(X)$.

In our case if ε is sufficiently small we can write the phase function $\Phi(x,y,\theta)$ in the conic neighborhood of the set C_Φ as

$$\Phi(x,y,\theta)=(x-y)\cdot\nabla_y\phi(y,\theta)+O(|x-y|^2|\theta|).$$

Introducing a new variable ξ by the relation

$$\xi=\nabla_y\phi(y,\theta),$$

we have

$$d\xi=h(y,\theta)\,d\theta,$$

where the function $h(y,\theta)$ is defined in Section 1, condition (iv).

It follows from condition (iv) and the explicit function theorem that there exists a function $\theta=\theta(y,\xi)$, $\theta(y,\xi)\in C^\infty(X\times\mathbb{R}^n)$, such that $\xi\equiv\nabla_y\phi(y,\theta(y,\xi))$ for $y\in X$. The function $\theta(y,\xi)$ is homogeneous of degree one with respect to ξ.

We can now rewrite the operator F_ε:

$$(F_\varepsilon u)(y) = \frac{1}{(2\pi)^n} \iint \exp\{i(x-y)\cdot\xi + O(|x-y|^2|\theta|)\} A_0(x, y, \xi) u(x)\, dx\, d\xi,$$

where

$$A_0(x, y, \xi) = \frac{A(x, y, \theta(y, \xi))}{h(y, \theta(y, \xi))} \chi_\varepsilon(x, y).$$

Both conditions of Theorem 3 are satisfied, if ε is sufficiently small. Hence, $F_\varepsilon \in \mathscr{L}^m(X)$. Since $\tilde{F} \in \mathscr{L}^{-\infty}(X)$, we see that $F \in \mathscr{L}^m(X)$.

To prove the second part of Theorem 2 we need the following

LEMMA 1. *An operator* $B \in \mathscr{L}^m(X)$, *where* $m < 0$, *can be extended to a compact operator from* $L^2(X, \text{compact})$ *to* $L^2(X, \text{loc})$.

The proof of Lemma 1 can be found in [18].

To complete the proof of Theorem 2 it is sufficient to show that $F - I \in \mathscr{L}^m(X)$, for some $m < 0$. We shall do this in the next section.

Let us summarize our results. Theorems 1 and 2 reduce the inversion problem for the generalized Radon transform (1.1) to solving a Fredholm integral equation. More precisely, given the generalized Radon transform $v(s, \omega) = (Ru)(s, \omega)$, we can find the function u as a solution of the integral equation

$$u + Tu = R^* K v, \tag{4.1}$$

where the operator T is given in Theorem 2.

The integral equation in (4.1) is a generalization of the well-known formula for the inversion of the classical Radon transform. It follows from the following

LEMMA. *If, in the conditions of the second part of Theorem* 2, *we assume in addition that*

$$\phi(x, \theta) = x\cdot\theta + \tilde{\phi}(\varepsilon x, \theta)$$

and

$$a(x, \theta) = \tilde{a}(\varepsilon x, \theta),$$

where ε *is a small parameter, then the operator* T *in* (4.1) *can be written as* $T = \varepsilon\tilde{T}$, *where* $\tilde{T}$ *remains bounded when* $\varepsilon \to 0$. *In particular, in the case of the classical Radon transform,* $T = 0$.

The proof of the lemma can be obtained by considering the operator $T = F - I$ and expanding its phase function and amplitude with respect to ε.

5. Asymptotic Expansion of the Operator F

In this section we construct the asymptotics of the operator F modulo regularizing operators. Thus, we can restrict our considerations to some conic neighborhood of the set C_Φ, where we can write the phase function $\Phi(x, y, \theta)$ in (2.2) as

$$\Phi(x, y, \theta) = \nabla_y \phi(y, \theta) \cdot (x - y) + H(x, y, \theta),$$

where

$$H(x, y, \theta) = \sum_{|\alpha|=2}^{\infty} \frac{1}{\alpha!} \partial_y^\alpha \phi(y, \theta)(x-y)^\alpha.$$

Here α is a multiindex, i.e., $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $\partial_y^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, where $\partial_k = (1/i)\partial/\partial y^k$.

We set

$$b(y, \theta) = \frac{h(y, \theta)}{a(y, \theta)},$$

and

$$U(|\theta|) = 1$$

in (2.1).

We consider the operator F_t:

$$(F_t u)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_X \exp\{i(x-y) \cdot \nabla_y \phi(y, \theta) + itH(x, y, \theta)\} \times \frac{a(x, \theta)}{a(y, \theta)} h(y, \theta) u(x)\, dx\, d\theta.$$

Obviously,

$$F_{t|_{t=1}} = F,$$

and we have

$$F = \sum_{m=0}^{N} \frac{1}{m!} \left(\frac{d}{dt}\right)^m F_{t|_{t=0}} + \int_0^1 \frac{(1-t)^N}{N!} \left(\frac{d}{dt}\right)^{N+1} F_t\, dt. \tag{5.1}$$

Introducing a new variable (see Section 4) by the relation $\xi = \nabla_y \phi(y, \theta)$, we obtain

$$(F_t u)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_X \exp\{i(x-y) \cdot \xi + itH(x, y, \theta(y, \xi))\} \frac{a(x, \theta(y, \xi))}{a(y, \theta(y, \xi))} u(x)\, dx\, d\xi,$$

where $\theta(y, \xi)$ is homogeneous of degree one with respect to ξ and $\xi \equiv \nabla_y \phi(y, \theta(y, \xi))$ for $y \in X$. We have

$$\left(\left(\frac{d}{dt}\right)^m F_t u\right)(y) = \frac{i^m}{(2\pi)^n} \int_{\mathbb{R}^n} \int_X \exp\{i(x-y) \cdot \xi + itH(x, y, \theta(y, \xi))\} \times H^m(x, y, \theta(y, \xi)) \frac{a(x, \theta(y, \xi))}{a(y, \theta(y, \xi))} u(x)\, dx\, d\xi. \tag{5.2}$$

The function $H^m(x, y, \theta)$ can be written in the form

$$H^m(x, y, \theta) = \sum_{|\alpha|=2m}^{|\alpha|=\infty} \frac{i^{|\alpha|}}{\alpha!} \prod_{k=1}^{m} \partial_y^{\alpha^k} \phi(y, \theta)(x-y)^{\alpha^k},$$

where $\alpha = (\alpha^1, \alpha^2, \cdots, \alpha^m)$, $|\alpha| = |\alpha^1| + \cdots + |\alpha^m|$, $\alpha! = \alpha^1! \cdots \alpha^m!$, $\partial^\alpha = \partial^{\alpha^1} \cdots \partial^{\alpha^m}$, and the element α^k is a multiindex, i.e., $\alpha^k = (\alpha_1^k, \cdots, \alpha_n^k)$, where $k = 1, 2, \cdots, m$.

Also, we have

$$\frac{a(x, \theta)}{a(y, \theta)} = \sum_{|\alpha^0|=0}^{|\alpha^0|=\infty} \frac{i^{|\alpha^0|}}{\alpha^0!} \frac{\partial_y^{\alpha^0} a(y, \theta)}{a(y, \theta)} (x-y)^{\alpha^0}.$$

The relation

$$\partial_\xi^{\alpha^k} e^{i(x-y)\cdot\xi} = (x-y)^{\alpha^k} e^{i(x-y)\cdot\xi},$$

where $k = 0, 1, \cdots, m$, implies that we can write (5.2) as

$$\left(\left(\frac{d}{dt}\right)^m F_t u\right)(y) = \frac{1}{(2\pi)^n} \sum_{|\alpha|=2m}^{|\alpha|=\infty} \frac{i^{|\alpha|+m}}{\alpha!}$$
$$\times \int_{\mathbb{R}^n} \int_X \partial_\xi^\alpha e^{i(x-y)\cdot\xi} \prod_{k=1}^{m} \partial_y^{\alpha^k} \phi(y, \theta) \frac{\partial_y^{\alpha^0} a(y, \theta)}{a(y, \theta)} e^{itH(x,y,\theta)} u(x)\, dx\, d\xi,$$

where $\theta = \theta(y, \xi)$ and $\alpha = (\alpha^0, \alpha^1, \cdots, \alpha^m)$. Here α^k is a multi index and $k = 0, 1, \cdots, m$.

Integrating by parts and setting $t = 0$ we obtain

$$\left(\left(\frac{d}{dt}\right)^m F_{t|_{t=0}} u\right)(y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_X e^{i(x-y)\cdot\xi}$$

$$\times \sum_{|\alpha|=2m}^{|\alpha|=\infty} \frac{(-i)^{|\alpha|-m}}{\alpha!} \partial_\xi^\alpha \left(\prod_{k=1}^{m} \partial_y^{\alpha^k} \phi(y, \theta) \frac{\partial_y^{\alpha^0} a(y, \theta)}{a(y, \theta)} \Bigg|_{\theta=\theta(y,\xi)} \right) u(x)\, dx\, d\xi. \tag{5.3}$$

Let T_l^m denote the operator defined by the formula

$$(T_l^m u)(y) = \frac{1}{(2\pi)^n} \sum_{|\alpha|=m+l} \frac{(-i)^l}{\alpha!} \int_{\mathbb{R}^n} \int_X e^{i(x-y)\cdot\xi}$$
$$\times \partial_\xi^\alpha \left(\prod_{k=1}^{m} \partial_y^{\alpha^k} \phi(y, \theta) \frac{\partial_y^{\alpha^0} a(y, \theta)}{a(y, \theta)} \Bigg|_{\theta=\theta(y,\xi)} \right) u(x)\, dx\, d\xi, \tag{5.4}$$

where $\alpha = (\alpha^0, \alpha^1, \cdots, \alpha^m)$.

The amplitude of the operator T_l^m is homogeneous of degree $-l = m - |\alpha|$ with respect to ξ. Hence, T_l^m belongs to $\mathscr{L}^{-l}(X)$. Since $\min(|\alpha|) = 2m$, we have

$$\left(\frac{d}{idt}\right)^m F_{t|_{t=0}} = \sum_{l=m}^{l=\infty} T_l^m,$$

and

$$\left(\frac{d}{idt}\right)^m F_{t|_{t=0}} \in \mathscr{L}^{-m}(X).$$

It is not difficult to see that Theorem 3 implies that an operator of the form (5.2) belongs to $\mathscr{L}^{-m}(X)$ for all $t\in[0,1]$. Using this fact and the relation (5.1) we have

$$F-\sum_{m=0}^{N}\frac{1}{m!}\left(\frac{d}{dt}\right)^m F_t\in\mathscr{L}^{-(N+1)}(X),$$

for $N=0,1,\cdots$.

Let us gather all operators with amplitudes of the same degree of homogeneity with respect to ξ. We obtain

$$F=\sum_{l=0}^{\infty}T_l \qquad \text{(up to a regularizing operator)}, \tag{5.5}$$

where

$$T_l=\sum_{m=0}^{m=l}T_l^m.$$

We compute now T_0 and T_1. Obviously, $T_0=I$. For T_1 we have

$$(T_1u)(y)=\frac{-i}{2(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\sum_{p,q=1}^{p,q=n}\frac{\partial^2}{\partial\xi^p\,\partial\xi^q}\left(\frac{\partial^2\phi(y,\theta)}{\partial y^p\,\partial y^q}\bigg|_{\theta=\theta(y,\xi)}\right)u(x)\,dx\,d\xi$$
$$+\frac{i}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\sum_{j=1}^{j=n}\frac{\partial}{\partial\xi^j}\left(\frac{\partial a(y,\theta)/\partial y^j}{a(y,\theta)}\bigg|_{\theta=\theta(y,\xi)}\right)u(x)\,dx\,d\xi.$$

Let us consider $h_{\nu\mu}(y,\theta)=\partial\xi^\nu/\partial\theta^\mu$ and $h^{\mu\nu}(y,\theta)=\partial\theta^\mu/\partial\xi^\nu$. Since $h_{\nu\mu}(y,\theta)=\partial^2\phi(y,\theta)/\partial y^\nu\partial\theta^\mu$ and $h_{\nu\mu}h^{\mu\nu'}=\delta_\nu^{\nu'}$, where $\delta_\nu^{\nu'}$ is the Kronecker symbol, we have

$$\sum_{q=1}^{q=n}\frac{\partial}{\partial\xi^q}\left(\frac{\partial^2\phi(y,\theta)}{\partial y^p\,\partial y^q}\bigg|_{\theta=\theta(y,\xi)}\right)=\frac{1}{h(y,\theta)}\frac{\partial}{\partial y^p}h(y,\theta)\bigg|_{\theta=\theta(y,\xi)},$$

where the function $h(y,\theta)$ is defined in Section 1, condition (iv). Hence, we obtain

(5.6)

$$(T_1u)(y)=\frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\sum_{j=1}^{j=n}\frac{1}{i}\frac{\partial}{\partial\xi^j}\left(\frac{\partial}{\partial y^j}\log\left(\frac{h^{1/2}(y,\theta)}{a(y,\theta)}\bigg|_{\theta=\theta(y,\xi)}\right)\right)u(x)\,dx\,d\xi.$$

We summarize our result as

THEOREM 4. *If $b(y,\theta)=h(y,\theta)/a(y,\theta)$ and $U(|\theta|)=1$ in (2.1), then (5.5) is the asymptotic expansion of the operator F, that is*

$$T_l\in\mathscr{L}^{-l}(X),\qquad l=0,1,\cdots,$$

and

$$\left(F-\sum_{l=0}^{N} T_l\right)\in \mathscr{L}^{-(N+1)}(X) \quad \text{for} \quad N=0,1,\cdots.$$

In particular,

$$(F-I)\in \mathscr{L}^{-1}(X).$$

Moreover, we have obtained the explicit formula (5.6) for the operator T_1.

COROLLARY 1. *If* $a(x,\omega)=h^{1/2}(x,\omega)$ *and, thereby,* $b(y,\omega)=h^{1/2}(y,\omega)$, *then* $(F-I)\in \mathscr{L}^{-2}(X)$.

Corollary 1 follows from (5.6).

6. The Generalized Radon Transform R_λ

The generalized Radon transform in (1.1) can formally be written as

$$(6.1)\qquad (R(a)u)(s,\omega)=\int_{\mathbb{R}^n} u(x)a(x,\omega)\delta(s-\phi(x,\omega))\,dx,$$

where δ denotes the δ-function concentrated on the surface $s=\phi(x,\omega)$. We use the notation $R(a)$ to indicate explicitly the density $a(x,\omega)$.

We introduce the generalized Radon transform $R_\lambda(a)$ as follows:

$$(6.2)\qquad (R_\lambda(a)u)(s,\omega)=\int_{\mathbb{R}^n} u(x)a(x,\omega)\frac{(s-\phi(x,\omega))_+^\lambda}{\Gamma(\lambda+1)}\,dx,$$

where $(s-\phi(x,\omega))_+=\max\{s-\phi(x,\omega),0\}$, and Γ is the Gamma function.

In general, we cannot define (6.2) for $[s,\omega]\in(\mathbb{R}\times S^{n-1})/Z_2$ as in (1.1). The transform (6.2) is defined for $(s,\omega)\in\mathbb{R}\times S^{n-1}$.

The parameter λ is complex-valued and the generalized Radon transform $R_\lambda(a)$ is an entire analytic function of λ. If $\lambda=-k$, where $k=1,2,\cdots$, then $R_{-k}(a)=\partial_s^{k-1}R(a)$. In particular, $R_{-1}(a)=R(a)$, where the transform $R(a)$ is described in (6.1).

The integral in (6.2) is well defined for $u\in C_0^\infty(X)$. We assume that the function $\phi(x,\omega)$ satisfies the properties (i)–(iv) and that the density $a(x,\omega)$ belongs to $C^\infty(X\times S^{n-1})$. We no longer assume in this and the following sections that the density $a(x,\omega)$ is positive and an even function with respect to ω.

However, the analogue of Theorem 1 holds if the densities $a(x,\omega)$ in (6.2) and $b(y,\omega)$ in (1.4) are either both even or both odd with respect to ω. In fact, we need their product $a(x,\omega)b(y,\omega)$ to be even with respect to ω to prove

THEOREM 5. *The Fourier integral operator F in* (2.1) *can be factored in the form*

$$F=R^*K_\lambda R_\lambda,$$

where R_λ is the generalized Radon transform (6.2), R^* is defined in (1.4) for $v \in C^\infty(R^1 \times S^{n-1})$, and K_λ is the convolutional operator with the generalized kernel

$$K_\lambda(s) = \frac{e^{-i(\lambda+1)\pi/2}}{2(2\pi)^n} \int_{-\infty}^{+\infty} |r|^{n-1} U(r)(r+i0)^{\lambda+1} e^{irs}\, dr. \tag{6.3}$$

To prove Theorem 5 we repeat the proof of Theorem 1 taking into account that

$$e^{-i(\lambda+1)\pi/2}(\xi+i0)^{\lambda+1} \int_{-\infty}^{+\infty} e^{is\xi}(R_\lambda(a)u)(s,\omega)\, ds = \int_{\mathbb{R}^n} a(x,\omega)\, e^{i\xi\phi(x,\omega)}\, u(x)\, dx.$$

The generalized Radon transform defined in (6.2) has the following property:

$$\partial_s R_\lambda(a) = R_{\lambda-1}(a), \tag{6.4}$$

which we shall use in the next section, where we discuss applications of the transform R_λ to partial differential equations.

7. Applications of the Generalized Radon Transform to Partial Differential Equations

It is well known that the classical Radon transform R_c reduces the Cauchy problem for the wave equation with $n+1$ independent variables

$$\begin{aligned}(\partial_t^2 - \Delta)u &= 0,\\ u(0,x) &= f_1(x),\\ u_t(0,x) &= f_2(x),\end{aligned} \tag{7.1}$$

to the problem with two independent variables

$$\begin{aligned}(\partial_t^2 - \partial_s^2)v &= 0,\\ v(0,s,\omega) &= (R_c f_1)(s,\omega),\\ v_t(0,s,\omega) &= (R_c f_2)(s,\omega),\end{aligned} \tag{7.2}$$

where

$$v(t,s,\omega) = (R_c u(t,x))(s,\omega).$$

(Let us recall that the classical Radon transform R_c is the special case of the generalized Radon transform (6.1), where $\phi(x,\omega) = x\cdot\omega$ and $a(x,\omega)=1$.)

For each ω, (7.2) is a solvable Cauchy problem in two independent variables, where ω is a parameter. The function $u(t,x)$ can be recovered by the inversion of the classical Radon transform (see [1], Chapter VI, Section 14, and [8]).

The generalized Radon transforms in (6.1) and (6.2) yield the same type of reduction for the case of equations with variable coefficients.

Let us consider the second-order elliptic differential operator with smooth coefficients

$$Lu = \partial_{x_i}(\alpha_{ij}\partial_{x_j}u) + \beta_i\partial_{x_i}u + \gamma u,$$

where $\alpha_{ij} = \alpha_{ji}$, $\tilde{\alpha}|\xi|^2 \leqq \alpha_{ij}\xi^i\xi^j \leqq \tilde{\beta}|\xi|^2$, $0 < \tilde{\alpha} \leqq \tilde{\beta}$, and the operator formally adjoint to it,

$$L^* v = \partial_{x_j}(\alpha_{ij}\partial_{x_i}v) - \partial_{x_i}(\beta_i v) + \gamma v.$$

We denote

$$[u, w] = \alpha_{ij}\partial_{x_i}u\partial_{x_j}w,$$

and

$$\tilde{L}w = \partial_{x_j}(\alpha_{ij}\partial_{x_i}w) - \beta_i\partial_{x_i}w.$$

LEMMA 2. *The composition of the generalized Radon transform in* (6.2) *and the differential operator L can be written as*

$$R_\lambda(a)L = R_{\lambda-2}(a[\phi, \phi]) - R_{\lambda-1}(a\tilde{L}\phi + 2[a, \phi]) + R_\lambda(L^* a). \tag{7.3}$$

To prove Lemma 2 we observe that we can differentiate the function $(s - \phi(x, \omega))_+^\lambda/\Gamma(\lambda + 1)$, i.e., the formal relation

$$\partial_{x_i}\frac{(s - \phi(x, \omega))_+^\lambda}{\Gamma(\lambda + 1)} = -\frac{(s - \phi(x, \omega))_+^{\lambda-1}}{\Gamma(\lambda)}\partial_{x_i}\phi(x, \omega)$$

is well defined (see [4], for example). Thus, we can prove Lemma 2 integrating the expression $R_\lambda(a)Lu$ by parts.

We would like to make use of the relation in (7.3) and thus our first step is to simplify it. Let $\phi(x, \omega)$ in (6.2) be the solution of the eikonal equation

$$[\phi, \phi] = 1, \tag{7.4}$$

such that $\phi(x, -\omega) = -\phi(x, \omega)$, where $\omega \in S^{n-1}$. We assume also that the function $\phi(x, \theta)$ satisfies conditions (i)–(iv), defining the function $\phi(x, \theta)$ for $\theta \neq 0$ as

$$\phi(x, \theta) = r\phi(x, \omega),$$

where $\theta = r\omega$ and $|\omega| = 1$. We assume here that such a solution exists in some domain X. (In general, the existence of this solution can be proven only locally.)

Let $a(x, \omega)$ be a positive solution of the transport equation

$$a\tilde{L}\phi + 2[a, \phi] = 0. \tag{7.5}$$

It follows from Lemma 2 and the relations (7.4), (7.5) and (6.4) that

$$R_\lambda(a)L = \partial_s^2 R_\lambda(a) + R_\lambda(L^* a). \tag{7.6}$$

We can consider now the Cauchy problem for the wave equation with variable coefficients:

$$(7.7)\qquad \begin{aligned}(\partial_t^2 - L)u &= 0,\\ u|_{t=0} &= f_1(x),\\ u_t|_{t=0} &= f_2(x),\end{aligned}$$

where $x \in X$. Making use of (7.6) we obtain

$$(7.8)\qquad \begin{aligned}(\partial_t^2 - \partial_s^2)v &= (R_\lambda(L^*a)u)(s, \omega),\\ v|_{t=0} &= (R_\lambda(a)f_1)(s, \omega),\\ v_t|_{t=0} &= (R_\lambda(a)f_2)(s, \omega),\end{aligned}$$

where

$$v(t, s, \omega) = (R_\lambda(a)u(t, x))(s, \omega).$$

The representation (7.8) is a generalization of (7.2) for the case of variable coefficients. It has a physical meaning since the function $a(x, \omega)$ is a solution of the transport equation (7.5). Let $a_k(x, \omega)$, $k = 0, 1, \cdots$, be a sequence of functions which satisfy the iterated transport equation

$$(7.9)\qquad \begin{aligned}a_0\tilde{L}\phi + 2[a_0, \phi] &= 0,\\ a_k\tilde{L}\phi + 2[a_k, \phi] &= L^*a_{k-1}, \qquad k = 1, 2, \cdots.\end{aligned}$$

In these notations $a(x, \omega) = a_0(x, \omega)$. We note that the density L^*a in (7.8) is the discrepancy in the iterated transport equation in (7.9). Equations in (7.9) are exactly the same as those which appear if we seek the solution of the Cauchy problem in (7.7) using the ray method.

We can emphasize such a connection by introducing the transform R_λ^N, $N = 0, 1, 2, \cdots$,

$$R_\lambda^N = \sum_{k=0}^{N} \partial_s^{-k} R_\lambda(a_k),$$

where the notation ∂_s^{-k} is used for the k-th antiderivative. The function $\phi(x, \omega)$ in the transforms $R_\lambda(a_k)$ satisfies the eikonal equation in (7.4) and is the same as in the previous case.

From Lemma 2 and relations (7.4), (7.9) and (6.4) we see that

$$(7.10)\qquad R_\lambda^N L = \partial_s^2 R_\lambda^N + \partial_s^{-N} R_\lambda(L^*a_N).$$

(Obviously, if $L^*a_{N^0} = 0$ for some N^0, then $R_\lambda^{N^0}L = \partial_s^2 R_\lambda^{N^0}$.)

For the Cauchy problem in (7.7) we have

$$(7.11)\qquad \begin{aligned}(\partial_t^2 - \partial_s^2)v &= \partial_s^{-N}(R_\lambda(L^*a_N)u)(s, \omega),\\ v|_{t=0} &= (R_\lambda^N f_1)(s, \omega),\\ v_t|_{t=0} &= (R_\lambda^N f_2)(s, \omega),\end{aligned}$$

where

$$v(t, s, \omega) = (R_\Lambda^N u(t, x))(s, \omega),$$

for $N = 0, 1, 2, \cdots$.

The relation in (7.10) means that the second-order elliptic differential operator L can be represented by the operator ∂_s^2 (up to a smoothing operator) in a domain, in which we can construct the appropriate generalized Radon transform (6.2). In this case the hypersurfaces are generated by the eikonal equation (7.4) and the densities $a_0, a_1, \cdots$ are solutions of the transport equations (7.9).

Although we shall not consider it here, we note that the representations (7.8) and (7.11) can be used for the construction of Riemann's radiation kernel for linear hyperbolic initial value problems in the case of variable coefficients (see [1], Chapter VI, Section 15).

Remark. We have chosen the second-order differential operator L for simplicity. The construction can be applied to higher-order differential operators as well.

8. The Inversion of the Exponential and Attenuated Radon Transforms

The exponential and attenuated Radon transforms arise in single photon emission tomography (see [17] for example). In this section we use the approach developed earlier in this paper to solve the inversion problem for these particular transforms.

We consider

$$(R_m u)(s, \omega) = \int_{\mathbb{R}^2} u(x)\, e^{m(x,\omega)} \delta(s - x \cdot \omega)\, dx, \tag{8.1}$$

where $e^{m(x,\omega)}$ is the density, ω is a unit vector and $m(x, \omega)$ is an infinitely differentiable function of its variables. The expression (8.1) is known as the attenuated Radon transform.

The inversion problem we address is the problem of reconstructing a function (which vanishes outside some bounded region X of the plane) from the known transform $(R_m u)(s, \omega)$ and a given function $m(x, \omega)$.

We also consider the exponential Radon transform

$$(R_\mu u)(s, \omega) = \int_{\mathbb{R}^2} u(x)\, e^{\mu x \cdot \omega^\perp} \delta(s - x \cdot \omega)\, dx, \tag{8.2}$$

where μ is a constant, and $\omega^\perp$ denotes the unit vector orthogonal to the vector $\omega = (\omega_1, \omega_2)$: $\omega^\perp = (-\omega_2, \omega_1)$. The transform in (8.2) is a special case of the attenuated Radon transform (8.1) corresponding to the case of the uniformly attenuating medium.

Since the function $m(x, \omega)$ does not necessarily equal $m(x, -\omega)$, we cannot define the attenuated Radon transform (8.1) on the space $(\mathbb{R} \times S^1)/Z_2$; thus we consider it on the cylinder $\mathbb{R} \times S^1$.

We define the even part of the transform R_m as follows

$$(R_m^{\text{even}}u)(s,\omega)=\int_{\mathbb{R}^2} u(x)a(x,\omega)\delta(s-x\cdot\omega)\,dx, \tag{8.3}$$

where

$$a(x,\omega)=\tfrac{1}{2}(e^{m(x,\omega)}+e^{m(x,-\omega)}).$$

Since $a(x,\omega)=a(x,-\omega)$, the transform (8.3) is defined on the space $(\mathbb{R}\times S^1)/Z_2$ and we can apply Theorems 1, 2 and 4.

We let the density of the dual transform be as is suggested in Theorem 2, i.e.,

$$(R_m^* v)(y)=\int_{|\omega|=1}\frac{1}{a(y,\omega)}v(s,\omega)|_{s=y\cdot\omega}\,d\omega. \tag{8.4}$$

By introducing the operator F in (2.1) for this case and applying Theorems 1 and 2 we reduce the problem of inversion of the attenuated Radon transform to solving the integral equation

$$u(y)+(T_m u)(y)=f(y), \tag{8.5}$$

where

$$f(y)=(R_m^* K v)(y).$$

Here T_m is the operator in Theorem 2 and the operator K has the generalized kernel

$$K(s)=\frac{1}{2(2\pi)^2}\int_{-\infty}^{+\infty}|r|\,e^{irs}\,dr. \tag{8.6}$$

The function

$$v(s,\omega)=\tfrac{1}{2}((R_m u)(s,\omega)+(R_m u)(-s,-\omega))$$

is given.

Theorem 4 provides us with the asymptotic expansion of the operator T_m.

The first term $\tilde{T}$ of the asymptotic expansion of the operator T_m (the most singular part) can be written as

$$(\tilde{T}u)(y)=\frac{i}{(2\pi)^2}\int_{\mathbb{R}^2}\int_X e^{i(x-y)\cdot\theta}\left(\sum_{k=1,2}\frac{\partial^2}{\partial\theta^k\,\partial y^k}\log a\left(y,\frac{\theta}{|\theta|}\right)\right)u(x)\,dx\,d\theta. \tag{8.7}$$

(It is possible to obtain more terms in the asymptotic expansion of the operator T_m if necessary.)

In the case of the exponential Radon transform in (8.2) we can derive the exact integral equation for the inversion problem.

We define transforms $(R_\mu^{\text{even}}u)(s,\omega)$ and $(R_\mu^{\text{odd}}u)(s,\omega)$ by the following relations:

$$(R_\mu^{\text{even}}u)(s,\omega)=\int_{\mathbb{R}^2} u(x)\cosh(\mu x\cdot\omega^\perp)\delta(s-x\cdot\omega)\,dx,$$

and

$$(R_\mu^{\text{odd}}u)(s,\omega)=\int_{\mathbb{R}^2} u(x)\sinh(\mu x\cdot\omega^\perp)\delta(s-x\cdot\omega)\,dx.$$

We introduce dual transforms $R_\mu^{*,\text{even}}$ and $R_\mu^{*,\text{odd}}$ as

$$(R_\mu^{*,\text{even}}v_+)(y)=\int_{|\omega|=1}\cosh(\mu y\cdot\omega^\perp)v_+(s,\omega)|_{s=y\cdot\omega}\,d\omega,$$

and

$$(R_\mu^{*,\text{odd}}v_-)(y)=\int_{|\omega|=1}\sinh(\mu y\cdot\omega^\perp)v_-(s,\omega)|_{s=y\cdot\omega}\,d\omega,$$

where the functions v_+ and v_- satisfy the relations $v_+(s,\omega)=v_+(-s,-\omega)$ and $v_-(s,\omega)=-v_-(-s,-\omega)$.

We consider the pseudo-differential operator F_μ,

$$(F_\mu u)(y)=\frac{1}{(2\pi)^2}\iint\cosh\left(\mu(x-y)\cdot\frac{\theta^\perp}{|\theta|}\right)e^{i(x-y)\cdot\theta}u(x)\,dx\,d\theta, \tag{8.8}$$

where $\theta^\perp$ is the vector orthogonal to the vector θ: $\theta^\perp=(-\theta_2,\theta_1)$. The operator F_μ can be written as $F_\mu=F_\mu^+-F_\mu^-$, where

$$(F_\mu^+u)(y)=\frac{1}{(2\pi)^2}\iint\cosh\left(\mu x\cdot\frac{\theta^\perp}{|\theta|}\right)\cosh\left(\mu y\cdot\frac{\theta^\perp}{|\theta|}\right)e^{i(x-y)\cdot\theta}u(x)\,dx\,d\theta,$$

and

$$(F_\mu^-u)(y)=\frac{1}{(2\pi)^2}\iint\sinh\left(\mu x\cdot\frac{\theta^\perp}{|\theta|}\right)\sinh\left(\mu y\cdot\frac{\theta^\perp}{|\theta|}\right)e^{i(x-y)\cdot\theta}u(x)\,dx\,d\theta.$$

It is easy to verify that Theorem 1 holds for the operators F_μ^+ and F_μ^-. Thus, we obtain

$$F_\mu^+=R_\mu^{*,\text{even}}KR_\mu^{\text{even}}$$

and

$$F_\mu^-=R_\mu^{*,\text{odd}}KR_\mu^{\text{odd}}.$$

The operator K has the generalized kernel described in (8.6).

We need the following

LEMMA 3.

$$I(\mu,x)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\left(\cosh\left(\mu x\cdot\frac{\xi^\perp}{|\xi|}\right)-1\right)e^{ix\cdot\xi}\,d\xi=\frac{\mu}{2\pi|x|}I_1(\mu|x|),$$

where I_1 is the modified Bessel function of order one.

The proof of Lemma 3 can be found in the appendix.

Lemma 3 implies that the operator F_μ in (8.8) can be represented as $F_\mu = I + T_\mu$, where the operator T_μ has the kernel

$$T_\mu(|x-y|) = \frac{\mu}{2\pi} \frac{I_1(\mu|x-y|)}{|x-y|}.$$

Thereby, the inversion problem for the exponential Radon transform is reduced to solving the integral equation

$$u(y) + \frac{\mu}{2\pi} \int_X \frac{I_1(\mu|x-y|)}{|x-y|} u(x)\, dx = f(y), \tag{8.9}$$

where

$$f(y) = (R_\mu^{*,\text{even}} K v_+)(y) - (R_\mu^{*,\text{odd}} K v_-)(y).$$

Functions v_+ and v_- are the even and odd components of the exponential Radon transform (8.2),

$$v_+(s, \omega) = \tfrac{1}{2}[(R_\mu u)(s, \omega) + (R_\mu u)(-s, -\omega)],$$

and

$$v_-(s, \omega) = \tfrac{1}{2}[(R_\mu u)(s, \omega) - (R_\mu u)(-s, -\omega)].$$

The functions v_+ and v_- are assumed given. It is easy to see from the power series expansion that the kernel of the operator T_μ is an infinitely differentiable function.

The integral equation (8.9) was first obtained by F. Natterer in [13] by a different approach. The numerical results of the inversion of the exponential Radon transform using the integral equation (8.9) can also be found in [13].

Appendix

To prove Lemma 3 we introduce the polar coordinates $\xi_1 = r\cos\psi$ and $\xi_2 = r\sin\psi$, such that

$$x\cdot\xi = r|x|\cos\psi,$$
$$x\cdot\xi^\perp = r|x|\sin\psi,$$

where $\xi = (\xi_1, \xi_2)$ and $\xi^\perp = (-\xi_2, \xi_1)$. We have

$$I(\mu, x) = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} (\cosh(\mu|x|\sin\psi) - 1)\, e^{ir|x|\cos\psi}\, d\psi\, r\, dr.$$

We set $y=\cos\psi$; then $\sin\psi=(1-y^2)^{1/2}$ for $\psi\in[0,\pi]$, and $\sin\psi=-(1-y^2)^{1/2}$ for $\psi\in[\pi,2\pi]$. We obtain

$$I(\mu,x)=\frac{1}{(2\pi)^2}\int_0^\infty\int_{-1}^{1}(\cosh(\mu|x|(1-y^2)^{1/2})-1)\,e^{ir|x|y}(1-y^2)^{1/2}\,dy\,r\,dr.$$

Expanding $\cosh(\mu|x|(1-y^2)^{1/2})$ in power series we have

$$I(\mu,x)=\frac{1}{(2\pi)^2}\int_0^\infty\sum_{k=1}^{\infty}\frac{\mu^{2k}|x|^{2k}}{(2k)!}\int_{-1}^{1}e^{ir|x|y}(1-y^2)^{k-1/2}\,dy\,r\,dr.$$

Since

$$J_k(z)=\frac{1}{\sqrt{\pi}\Gamma(k+\frac{1}{2})}(\tfrac{1}{2}z)^k\int_{-1}^{1}e^{izy}(1-y^2)^{k-1/2}\,dy,$$

where J_k is the Bessel function of order k, we obtain

$$I(\mu,x)=\frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{\mu^{2k}|x|^k}{2^k k!}\int_0^\infty r^{-k+1}J_k(r|x|)\,dr.$$

We have (see [9], formula 21.8-23)

$$\int_0^\infty r^{-k+1}J_k(ar)\,dr=2^{1-k}a^{k-2}\frac{1}{(k-1)!},$$

where $k\geqq 1$. Hence,

$$I(\mu,x)=\frac{\mu}{2\pi|x|}\sum_{k=1}^{\infty}\frac{1}{k!(k-1)!}\left(\frac{\mu|x|}{2}\right)^{2k-1},$$

and the proof is completed.

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